Smith.

On The Projective Methods For Investigating The Singularities At Infinity Of Curves

ON THE PROJECTIVE METHODS FOR INVESTIGATING THE SINGULARITIES AT INFINITY OF CURVES

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I HEREBY RECOMMEND THAT THE THESIS PREPARED UNDER MY SUPERVISION BY Merlin Grant Smith

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TABLE OF CONTENTS.

CHAPTER I.

INTRODUCTION

CHAPTER II.

CONICS AND CIRCULAR CURVES.

CHAPTER III.

ALGEBRAIC AND TRANSCENDENTAL CURVES.

CHAPTER IV.

BINOMIAL CURVES.

13. $y = ax^n$.

14. $y = ax^{p/q}$.

CHAPTER V.

CYCLIC CURVES.

15. $(x^2 + y^2)^k \varphi_{n-2k}(x,y) + \psi_{n-p}(x,y) = 0.$

16. Lemniscate.

17. Pascal's Limaçon.

18. Cardioid.

19. Three-cusped Hypocycloid.

20. Three-phase Pendulum.

21. $[(x-1) f(x,y) + (y^2 - 1) g(x,y)]^2 + \lambda (x-1)^n h(x,y) = 0.$

22. $x^3 - 3xy^2 - a^3 = 0.$

23. Conclusion.

ON THE PROJECTIVE METHODS FOR INVESTIGATING THE SINGULARITIES AT INFINITY OF CURVES

CHAPTER I.

INTRODUCTION

1. Historical Note. It is evident from the ancient Egyptian statues that the methods of projection were entirely unknown to the geometers of the time. It was left to the Greeks, in fact to Anaxogoras and Democritus to discover the laws of a vanishing point. Until the fifteenth century little use was made of this discovery, but at this time Jan van Eyck utilized the laws of perspective in the great altar painting at Ghent. During this century and the following two, Albrecht Dürer, J. Cousin, Piero and Guido Ubaldi arrived at some important results in making perspective drawings and in the laws of the vanishing points of systems of parallel lines. What these men simply foreshadow Simon Stevin clearly grasps in its principal features and in an important theorem lays the foundation for the development of the theory of collineation. In the latter part of the eighteenth century Newton published a book containing an exhaustive study of a cubic equation and in this work he made the discoveries which led to the development of the analytic theory of perspective. In discussing the multiple points of a curve at infinity and in the finite region he introduced "Newtons parallelogram" or "analytic triangle." These results are based on the conception due to Desargues that two parallel lines meet at infinity.

At the beginning of the nineteenth century Möbius introduced homogeneous coordinates and also established the one to one correspondence of figures in collineation. In 1839, Plücker published a volume which gave the equations of the analytic conditions for certain singular points of fourth order curves. He also classified all third order plane curves by the nature of the singularities at infinity. Cayley extended these singularities to singularities of higher order and derived "equivalence numbers" which enable us to determine how many singularities are absorbed into a singular point of higher order and how the expression for the deficiency of the curves is modified thereby. How a curve of elementary singularities can be derived from a curve of higher singularities for which the Plücker and deficiency equations are the same was studied by A. Brill, Clebsch and others.

2. The Problem. Since the character of a curve at the infinitely distant points cannot be studied by the ordinary means at our disposal, several methods have been devised whereby such a study can be made. It is evident at once that, in order to investigate the singularities at infinity of a curve, the infinite points must be projected to the finite plane. Our problem then consists in the discussion of several methods by which the infinite points of a curve are transformed to the finite plane and in such a way that the form of the curve at the point in question is preserved. When the character of the curve at the projected infinite point is determined then we have established the character of the singularity at infinity. No essentially new methods are proposed in this paper but the details of the constructions and analytical proofs of the results are given since in most cases they are found discussed

in only a general way.

3. General Analytic Expression for Perspective. A clear geometrical conception of a perspective is necessary in order that the analytic expressions involved may be more tangible. Consider a plane P, in which we may have a curved or straight line described by the motion of a point A in the plane. Consider also a point C outside of this plane and besides a second plane P' which does not contain C but which intersects the plane P along the line s. If we now have the line from C through A, extended so that it intersects the plane P' at A', as the point A traces out the curve in the P plane the point A' describes a curve in the P' plane. The latter is called the perspective of the former or vice versa.

Since our study for the most part involves the character of the curves at the infinitly distant points of the plane P, we shall develop the geometrical concept of such a perspective and then give a perspective transformation and show how it fulfills the conditions necessitated by our geometrical concept. Let us take a plane Q through C parallel to P. By a convention of projective geometry the plane Q intersects the plane P along the infinitely distant line. Since the Q plane may be considered as formed by the lines from the infinitely distant points of P passing through C and since these lines all intersect the P' plane they must intersect it along the line q' which is the intersection of the P' plane and the Q plane. Therefore the infinitely distant points of P are projected into the line q' in the P' plane. In a like manner it can be shown that the points of the line formed by the intersection of the plane P and a plane through C parallel to P' denoted by R are projected to the infinity distant points of the P' plane.

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For convenience in showing the relation between the original curve and the projected curve we shall assume that each of the planes of our space figure given above is rotated about the line s, with the plane P fixed, until all the planes coincide. The point C lies at a point of P on a line formed by the intersection of P and a plane through C normal to s.

Let us take the equations of our perspective in this form,

$$(1)\qquad x' = \frac{x}{ax + by + c}, \qquad y' = \frac{y}{ax + by + c},$$

where x and y are the coordinates of the points in the P plane while x' and y' are the coordinates of the points in the plane P'.

It is evident from our geometrical interpretation that the line s is invariant, i.e., the values of x and y are equal to the values of x' and y' at the corresponding points on the line. From this we have $x = x'$ and $y = y'$. Therefore substituting in (1) and simplifying we have

$$ax + by + c = o$$

as the equation of the invariant line s.

To find the points of P which are transformed to infinity we substitute those values of x and y which will make x' and y' infinite. Indeed we shall put

$$ax + by + c = o \quad \text{where } x \neq o \text{ and } y \neq o.$$

This line which we have denoted by r is parallel to s. It is clear then that the points of a curve in the P plane which coincide with this line are transformed to infinity.

We next desire to find the locus of points which are the projections of the infinite points of P. If we solve (1) for x and y in terms of x' and y' and then replace the denominator by zero when the numerator is different from zero we shall have the required condition. From (1)

$$ax'x + bx'y + cx' - x = o\,,$$

$$ay'x + by'y + cy' - y = o\,,$$

and

$$(2) \qquad x = \frac{cx'}{-ax' - by' + 1}\,,$$

$$y = \frac{cy'}{-ax' - by' + 1}\,. \qquad c \neq o$$

Now $ax' + by' - 1 = o$ is the locus of points desired and is a line parallel to s denoted by q'.

We have now the three lines fundamental in our construction of projective curves. If we take C at the origin we can easily verify that the distance from C to q' is equal to the distance between r and s and the distance from C to any of these lines is a constant. From this we can show that our projection satisfies the condition for the absolute invariant of projection.

In Fig. 1[1] we see that

$$(CTAA') = (CURV)\,,$$

$$\frac{CA}{TA}\cdot\frac{TA'}{CA'} = \frac{CR}{UR}\cdot\frac{UV}{CV}\,,$$

$$\lim_{UV \doteq \infty}\left(\frac{UV}{CV}\right) = \lim_{UV \doteq \infty}\left(\frac{UV}{CU + UV}\right) =$$

$$\lim_{UV \doteq \infty}\left(\frac{1}{\frac{CU}{UV} + 1}\right) = 1\,.$$

1. See Karl Doehlemann - Geometrische Transformation. Vol. I, pp. (166 - 170).

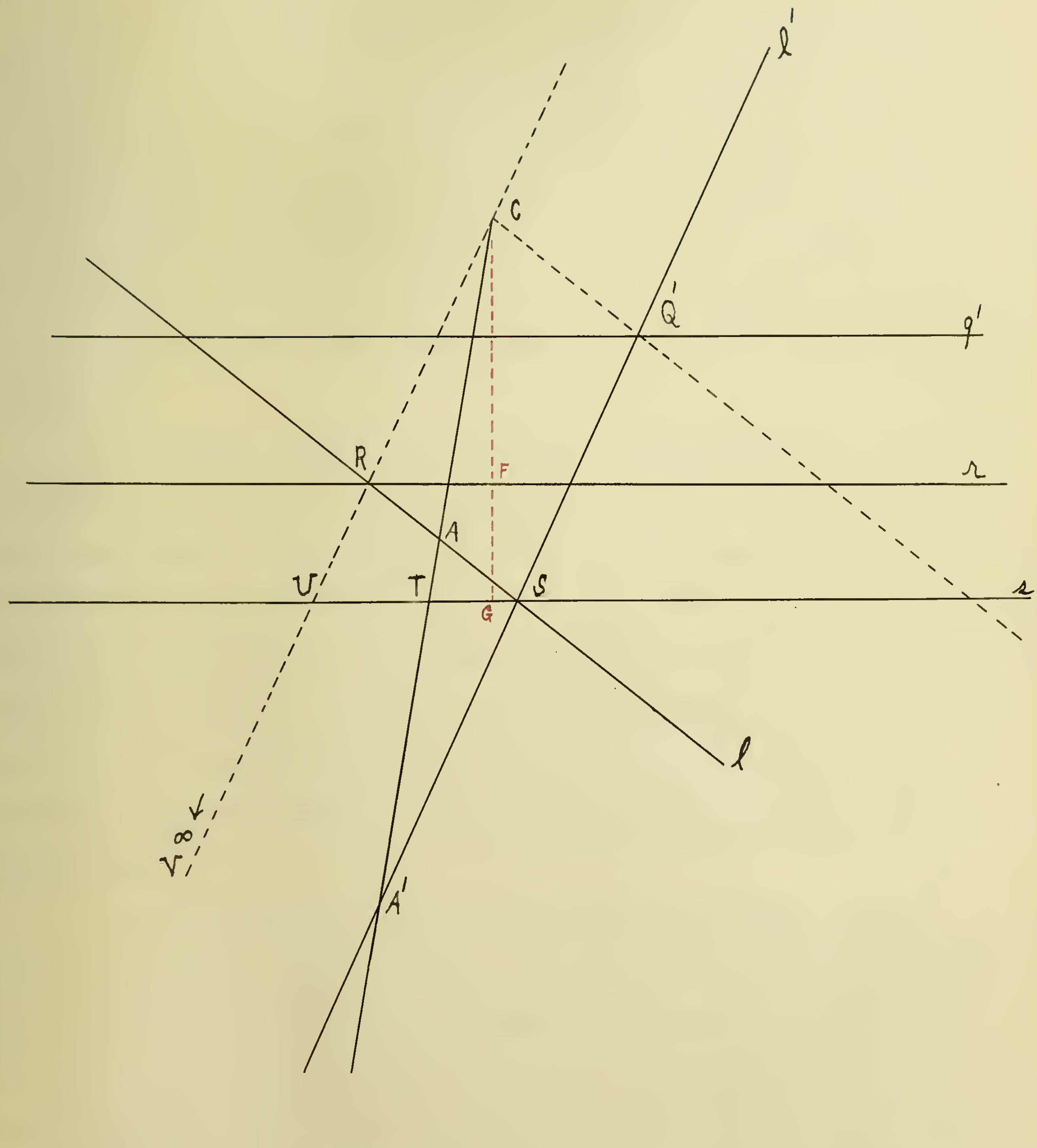

Fig. 1.

Substitute above,

$$\frac{CR}{UR} \cdot \frac{UV}{CV} = \frac{CR}{UR} = \frac{CF}{GF} = \text{const.}$$

We have now established the conditions whereby we may study the character of the infitely distant points of a curve. It is already well known that such a transformation does not alter the character of the curve in the neighborhood of a point. Since this is true, after we have transformed our curve we can examine its character at the points coincident with q' and the nature of the curve at these points will determine the form of the curve at infinity.

4. General Method of Homogeneous Coordinates. We have just given one method by which the infinitely distant points of a curve are transformed to the finite portion of the plane. Several methods have been devised for the study of curves at their infinitely distant points and would satisfy our purpose as long as the character of the curve was not changed by the given transformation. We shall however give only one more method, one which in certain cases is more convenient for our purpose. This is the method of homogeneous coordinates.[1]

Let us consider the ordinary coordinate axes x and y. Now if we join the infinite point of the x axis with the infinite point of the y axis we have an infinitely distant line which together with the two axes form a triangle. Now let us choose a triangle wholly within the finite region and associate with its sides the x axis, the y axis and the infinitely distant line

(1) See W. Killing--Lehrbuch der analytischen Geometrie in Homogenen Koordinaten. 1. Teil: Die Ebene Geometrie.

respectively. Moreover we have by doing so changed the infinitely distant points of the original plane to the finite plane in such a way that we can investigate our curve in the neighborhood of the line indicated and thus know its properties at infinity. A complete discussion of the properties of this triangle and a proof of the one to one correspondence of the integers with the points of the triangle will be found in the notes prepared by Dr. Arnold Emch at the University of Illinois for the class in Elements of Projective Geometry; also the Geometrischen Transformationen by Karl Doehlemann. We shall give briefly a discussion of those properties of the homogeneous triangle which are useful for our purpose.

In order to change from the rectangular coordinates to the homogeneous coordinates we substitute the following values:

$$x = \frac{x_1}{x_3}, \qquad y = \frac{x_2}{x_3}.$$

The equation of the curve then assumes the form

$$f(x_1, x_2, x_3) = 0$$

which can be plotted on the triangle selected. The equation of the tangent to the curve is

$$\frac{\partial f}{\partial x_1} y_1 + \frac{\partial f}{\partial x_2} y_2 + \frac{\partial f}{\partial x_2} y_3 = 0,$$

where y_1 y_2 and y_3 are the current coordinates of the point on the curve. If now the values of y_1 y_2 and y_3, where the curve intersects the line of our triangle corresponding to the infinitely distant line, are substituted in the above equation we obtain the equation of the tangent at the infinitely distant point of the curve. For the cases considered in this treatise the line above is always tangent to the curve at the point. The character of the

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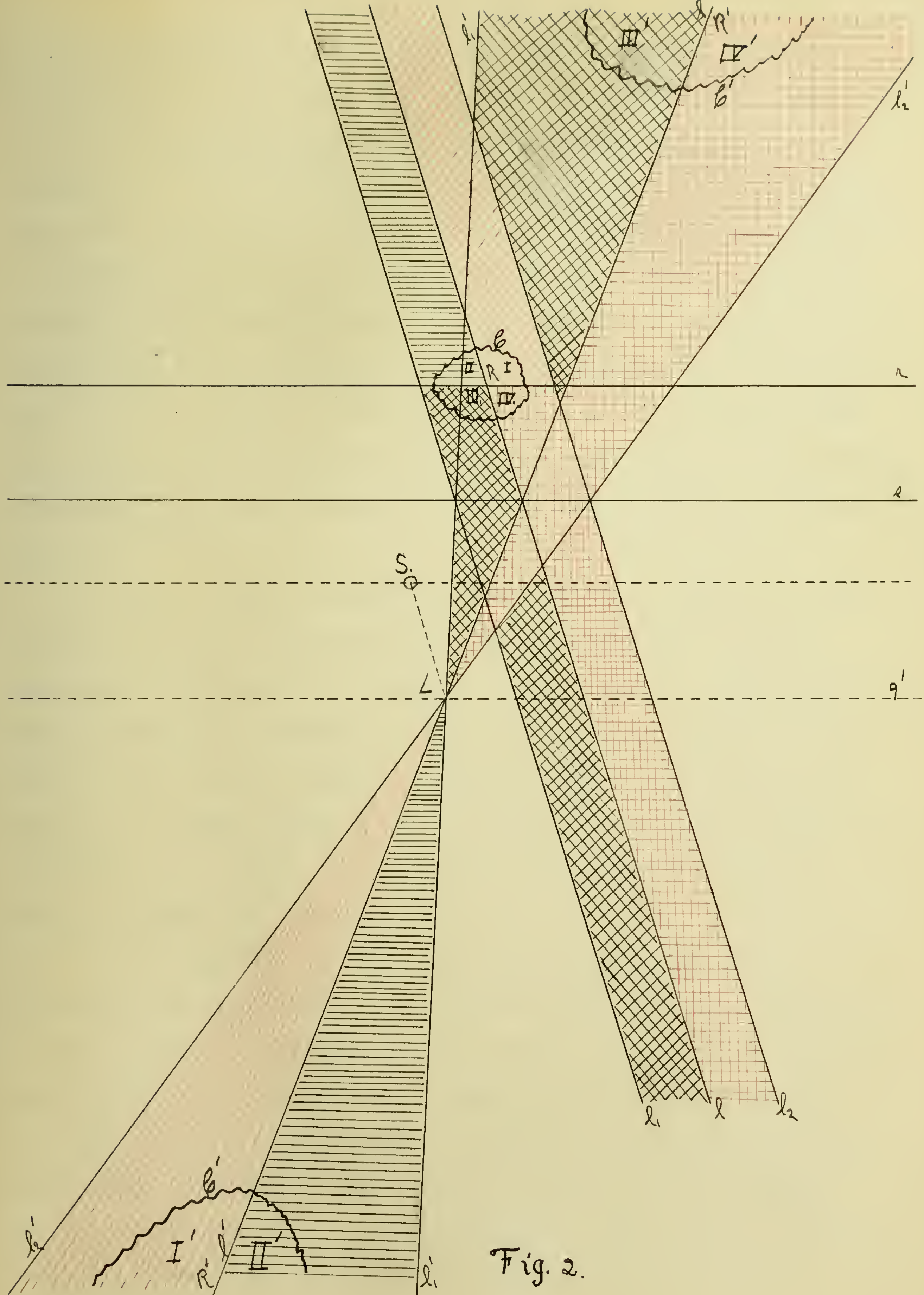

Fig. 2.

in the neighborhood of the origin is preserved and we can see, by plotting the curve, the path by which it becomes infinite.

5. Transformation of the Region about a Point. To investigate the form, which the singularities of a curve at a point take, when the point is transformed by a perspective transformation to a point at infinity, we will consider the regions, into which the portions of the plane in the neighborhood of the point bounded by given lines, are transformed by such a transformation. In Fig. 2 take S as the center of projection, then s is the invariant line and r and q' the lines representing the projections of the infinite portions of the planes determined by the lines q's and rs respectively.

Given in the rs plane the three parallel lines l, l_1 and l_2, and the point R determined by the intersection of l and r about which the region for investigation is taken. Consider the region within C as divided up into four quadrants formed by the lines l and r and the curve C. However before restrictimg the region to so small and area let us examine the regions bounded by the lines l, l_1 and r also l, l_2 and r, regarding the regions above and below r as distinct.

It is easily seen that when the construction of the transformed configuration is made that l goes into l', l_1 into l_1', l_2 into l_2' and the infinite point of each of these lines into the intersection of l' and q' or L. Since the line s is invariant the regions below r bounded by l, l_1 and l_2 are transformed into those above L bounded by the lines l', l_1' and l_2'; the regions above r are transformed to those below L. The point R is transformed to the infinite point of the line l' and the regions

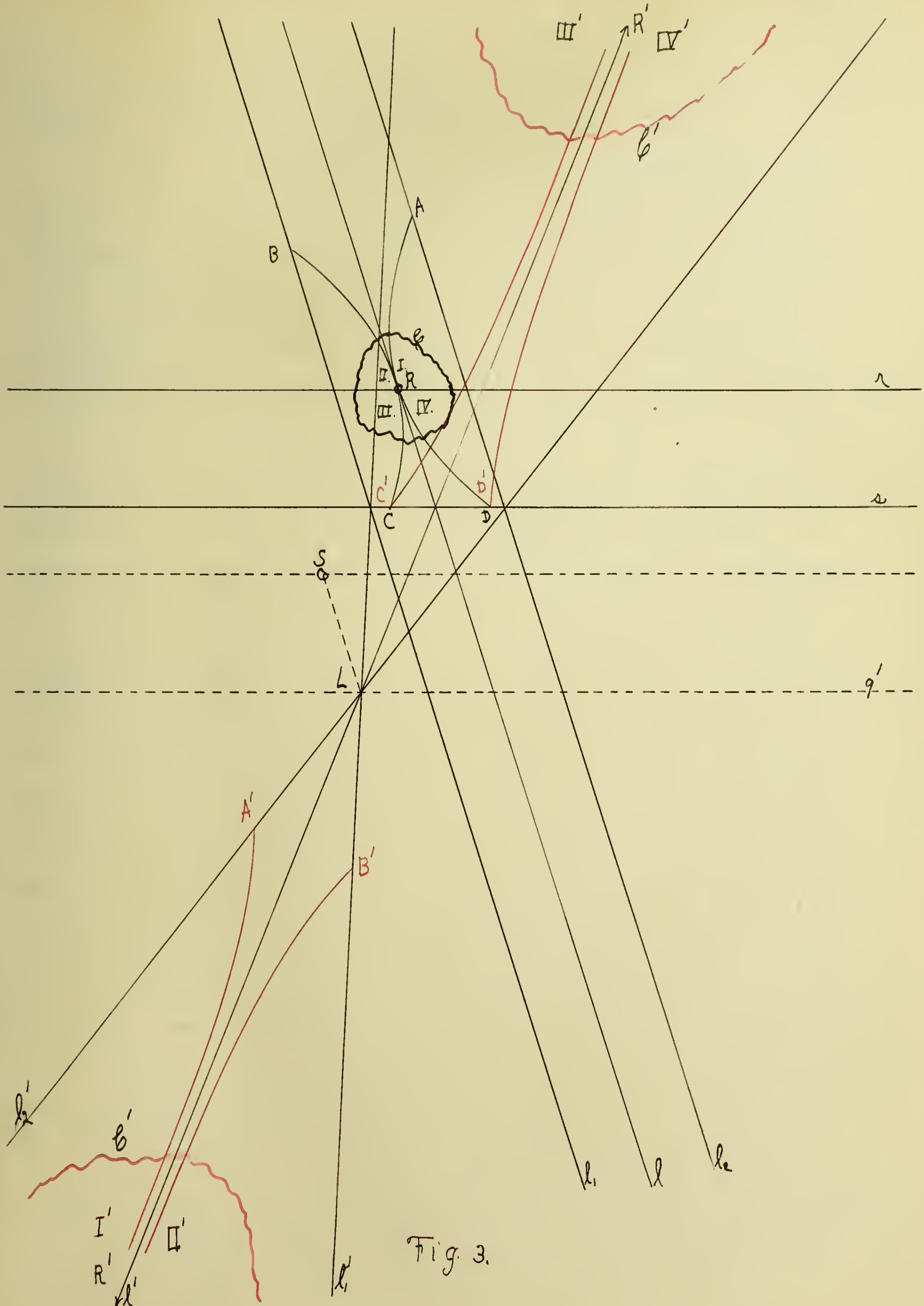

Fig. 3.

within C denoted by 1, 11, 111 and 1V go over into regions within C' about R' denoted by 1',11', 111' and 1V' respectively.

Thus to accomplish the purpose of the problem we need only to investigate the form of the transformed curve in the neighborhood of R'. Two general cases arrive; i.e. when the curve has l as the tangent at the singular point and when it has r as the tangent at the point. The former will be considered first.

In Fig. 3, for convenience, draw the curves so that there are two curves ARB and CRD with cusps at R, two curves ARC and BRD with points of inflexion at R, and two curves ARD and BRC with ordinary tangent points at R. The curves are transformed into curves having their singularities at R' and with l' as a tangent. It is evident that ARB and CRD go over into A'R'B' and C'R'D' respectively. A curve with a cusp is then transformed into two branches asymptotic to the transformed tangent line, lying on different sides of it and coming in the same direction from the infinite point. This description refers to a cusp of the first kind but applies equally well for one of the second kind except that both branches will be on the same side of the tangent line or asymptote. Now ARC and BRD are transformed into A'R'C' and B'R'D' respectively. A curve with a point of inflexion is transformed into two branches asymptotic to the transformed tangent line, lying on the same side of it and coming in opposite directions from the infinite point. Finally ARD and BRC go over into A'R'D' and B'R'C' respectively. It follows then that a curve with an ordinary tangent point is transformed into two branches asymptotic to the transformed tangent line, lying on opposite sides of it and coming in opposite directions from the infinite point.

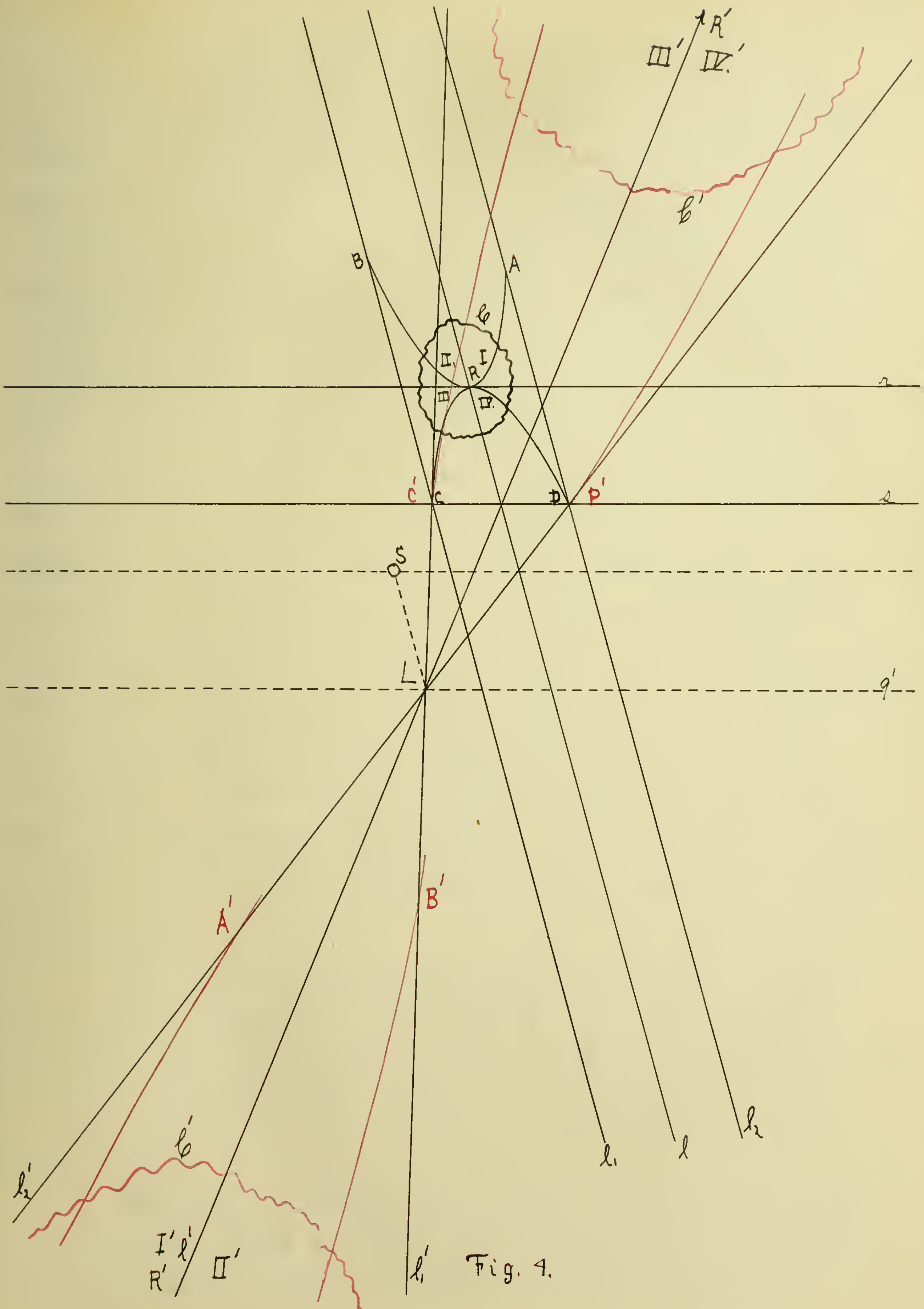

Fig. 4.

Now consider the case where there are the same three kinds of singularities, each represented by two different curves but having r as the tangent line at the point R. The tangent line for the transformed curves is the infinite line of the plane q's. In Fig. 4 ARD and BRC are transformed into A'R'D' and B'R'C' respectively. A curve with a cusp is then transformed into two branches both tangent to the infinite line, lying on opposite sides of a line passing through L and R' denoted by l' and coming in opposite directions from the infinite point. For a cusp of the second kind both branches would be on the same side of l' and come in the same direction from the infinite point. The curves ARC and BRD go over into A'R'C' and B'R'D' respectively. It is clear then that a curve with a point of inflexion is transformed into two branches, both tangent to the infinite line, lying on the same side of l' and coming in opposite directions from the infinite point. Lastly ARB and CRD are changed into A'R'B' and C'R'D' respectively. A curve with an ordinary tangent point is transformed into two branches, both tangent to the infinite line, lying on opposite sides of l' and coming in the same direction from the infinite point.

Since every singularity may be decomposed into one or more of these singularities just studied, it follows that the method used here may be applied equally as well to other singularities. Consequently the method applies equally well to all singularities.

The first two methods apply to the analytical discussion of the projected curve and one or both of them will be used for every curve which is studied. The last method is however a

construction method. This will be used for every curve which has a definite form. The pure black curve will represent the original curve while the transformed curve will be given in red. We are now in a position to apply our methods to various forms of curves. No attempt is made that this should be an exhaustive study, but a few examples are selected from the curves met with most often and the results for these computed.

CHAPTER II.

CONICS AND CIRCULAR CURVES.

6, $x^2+y^2-\lambda = 0$. The following problem furnishes an example of the application of the first method. Given a concentric pencil of circles

$$x^2 + y^2 - \lambda = 0,$$

λ being the variable parameter. Prove that this pencil is projected into a pencil of conics which are all tangent to each other at two fixed points of q'.

Assuming the perspective transformation as given in equation (2) # 2.

$$(1) \quad x = \frac{cx'}{-ax'-by'+1}, \qquad y = \frac{cy'}{-ax'-by'+1},$$

and substituting these values in

$$x^2 + y^2 - \lambda = 0 ,$$

the resulting equation is

$$(2) \quad \frac{c^2x'^2}{(-ax'-by'+1)^2} + \frac{c^2y'^2}{(-ax-by'+1)^2} - \lambda = 0,$$

which when simplified gives

$$(3) \quad c^2x'^2 + c^2y'^2 - \lambda a^2x'^2 - \lambda b^2y'^2 - 2\lambda abx'y' + 2\lambda ax' + 2\lambda by' - \lambda = 0,$$

This is the most general form of the equation of a conic . Hence the first part of the problem is proved.

To determine the points of intersection of the conics with q', the equation of q' from # 2 is solved simultaneously with the conics (3).

$$(4) \quad ax' + by' - 1 = 0,$$

from which

$$(5) \qquad x' = \frac{1 - by'}{a}, \qquad y' = \frac{1 - ax'}{b}.$$

Substituting the second of these in (3) and reducing the result, the equation is a quadratic in x', thus

$$(6) \qquad x'^2(a^2+b^2) - 2ax' + 1 = 0,$$

From this equation

$$(7) \qquad x' = \frac{a \pm \sqrt{a^2 - (a^2 + b^2)}}{a^2 + b^2} = \frac{a \pm ib}{a^2 + b^2},$$

$$(8) \qquad x' = \frac{1}{a-ib} \quad \text{or} \quad \frac{1}{a+ib}.$$

In a similar way, by substituting the first of equation (5) in (3) there results the quadratic equation in y'

$$(9) \qquad y^2(a^2+b^2) - 2by' + 1 = 0$$

from which

$$(10) \qquad y' = \frac{b \pm \sqrt{b^2-(a^2+b^2)}}{a^2+b^2} = \frac{b \pm ia}{a^2+b^2},$$

$$(11) \qquad y' = \frac{1}{b-ia} \quad \text{or} \quad \frac{1}{b+ia}.$$

These values of x' and y' could be associated in such a way as to represent four points, but since a straight line can intersect a conic in only two points, and since these values of x' and y' are independent of the parameter λ it is evident that there are only two points on q' through which all the conics pass. Since the values of x' and y' are complex these points are imaginary and the points of which they are the projections are called the <u>circular points at infinity.</u>

It still remains to be shown that they are tangent to each other at these points. The slope of the curves at any point is given by

$$(12) \qquad \frac{dy'}{dx'} = -\frac{2c^2x' - 2\lambda a^2x' - 2\lambda aby' - 2\lambda a}{2c^2y' - 2\lambda b^2y' - 2\lambda abx' - 2\lambda b}.$$

It is found by substituting in (12)

(13) $x' = \frac{1}{a-ib}$, $y' = \frac{1}{b+ia}$

there results

(14) $\frac{dy'}{dx'} = -\frac{a+ib}{b-ia}$.

Also when the remaining values of x' and y', i.e.

(15) $x' = \frac{1}{a+ib}$, $y' = \frac{1}{b-ia}$

are substituted in (12) we have

(16) $\frac{dy'}{dx'} = -\frac{a-ib}{b+ia}$.

Since the values of $\frac{dy'}{dx'}$ at the points

$$\left(\frac{1}{a-ib}, \frac{1}{b+ia}\right) \text{ and } \left(\frac{1}{a+ib}, \frac{1}{b-ia}\right)$$

are both independent of the parameter λ it follows that the curves are tangent to the same line and hence are tangent to each other at the points I_1', and I_2'. From this it follows that the circles of a concentric pencil may be considered as tangent to each other and to the line at infinity at the circular points I_1 and I_2 of which I_1' and I_2' are the perspectives.

From this discussion it is evident that every curve has some points at infinity either real or imaginary. Whether the points are real or imaginary we can investigate the properties of the curve in the neighborhood of the points by transforming it as above and then determining the form of the transformed curve where it intersects q'. Since the method may be applied to both real and imaginary infinitely distant points of a curve it is a general method and useful therefore in discussing the properties of any curve.

7. <u>Discussion of a Curve of the form</u> $(x' - x_1')(x' - x_2')\cdot$ $\phi(x'\,y') + (y' - y_1')(y' - y_2')\,\psi(x'\,y') = 0$.

The problem which was just considered shows that the circles of

the form $x^2+y^2=\lambda$ and in fact all circles pass through the circular points at infinity. I shall next establish the general form of a class of algebraic curves that pass through the circular points. The transformed curve may be written in the form

$$(1)\quad (x'-x_1')(x'-x_2')\,\varphi(x',y') + (y'-y_1')(y'-y_2')\,\psi(x',y') = 0,$$

where $\varphi(x'\ y')$ and $\psi(x'\ y')$ are real functions of x' and y' and are of degree m and n respectively where $m > n$ and

$$(2)\quad I_1'\begin{cases} x' = \dfrac{a+ib}{a^2+b^2}, \\[2ex] y' = \dfrac{b-ia}{a^2+b^2}, \end{cases} \qquad I_2'\begin{cases} x' = \dfrac{a-ib}{a^2+b^2}, \\[2ex] y' = \dfrac{b+ia}{a^2+b^2}. \end{cases}$$

This curve is real for if the values for x_1', x_2' and y_2' are substituted the resulting equation has the form

$$(3)\quad \left(x'^2 - \frac{2ax'}{a^2+b^2}+1\right)\varphi(x',y') + \left(y'^2 - \frac{2by'}{a^2+b^2}+1\right)\psi(x',y') = 0,$$

where the imaginary parts have disappeared. If x_2' and y_2' or x_2' and y_1' or x_1' and y_2' had been interchanged the imaginary parts of the equation would have remained and the curve would not have been real. The curve is therefore real and passes through the points I_1', and I_2'.

The projection from the P' plane to the P plane is made by the transformation

$$(4)\quad x' = \frac{x}{ax+by+c}, \qquad y' = \frac{y}{ax+by+c}.$$

After transforming the curve by substituting (4) in (2) we obtain

$$(5)\quad \left[\frac{x^2}{(ax+by+c)^2} - \frac{2ax}{(ax+by+c)(a^2+b^2)} + \frac{1}{a^2+b^2}\right]\frac{\varphi(x,y)}{(ax+by+c)^m}$$
$$+\left[\frac{y^2}{(ax+by+c)^2} - \frac{2by}{(ax+by+c)(a^2+b^2)} + \frac{1}{a^2+b^2}\right]\frac{\psi(x,y)}{(ax+by+c)^n} = 0.$$

Now multiplying through by $(ax+by+c)^{m+2}\ (a^2+b^2)$ and simplifying the result we obtain an equation which may be put in the form

$$(6)\quad (x^2+y^2)\,F(x,y) + G(x,y) = 0,$$

where the second term is of degree at least one less than the first term. Now dividing through by x^{m+2} where m is the degree of F (x,y) the equation has the form

$$(7) \quad \left[1 + \left(\frac{y}{x}\right)^2\right] \frac{F(x,y)}{x^m} + \frac{G(x,y)}{x^{m+2}} = 0.$$

Passing to the limit as x becomes infinite we have

$$(8) \quad (1 + i\,\frac{y}{x})\,(1 - i\,\frac{y}{x}) = 0,$$

or

$$(9) \quad x^2 + y^2 = 0 .$$

From (8) it is evident that the slope of the curve for the circular points is + i or - i respectively.

From this discussion it is clear that every curve passing through the points I_1', and I_2' will after a transformation similar to the one above contain a factor of the form (9). On the other hand evidently a curve having a factor of this form (1) passes through the cyclic points and from this we have a distinguishing property of such curves which is readily recognizable from their general form.

(1) For a further discussion of curves of this form see Chap. V.

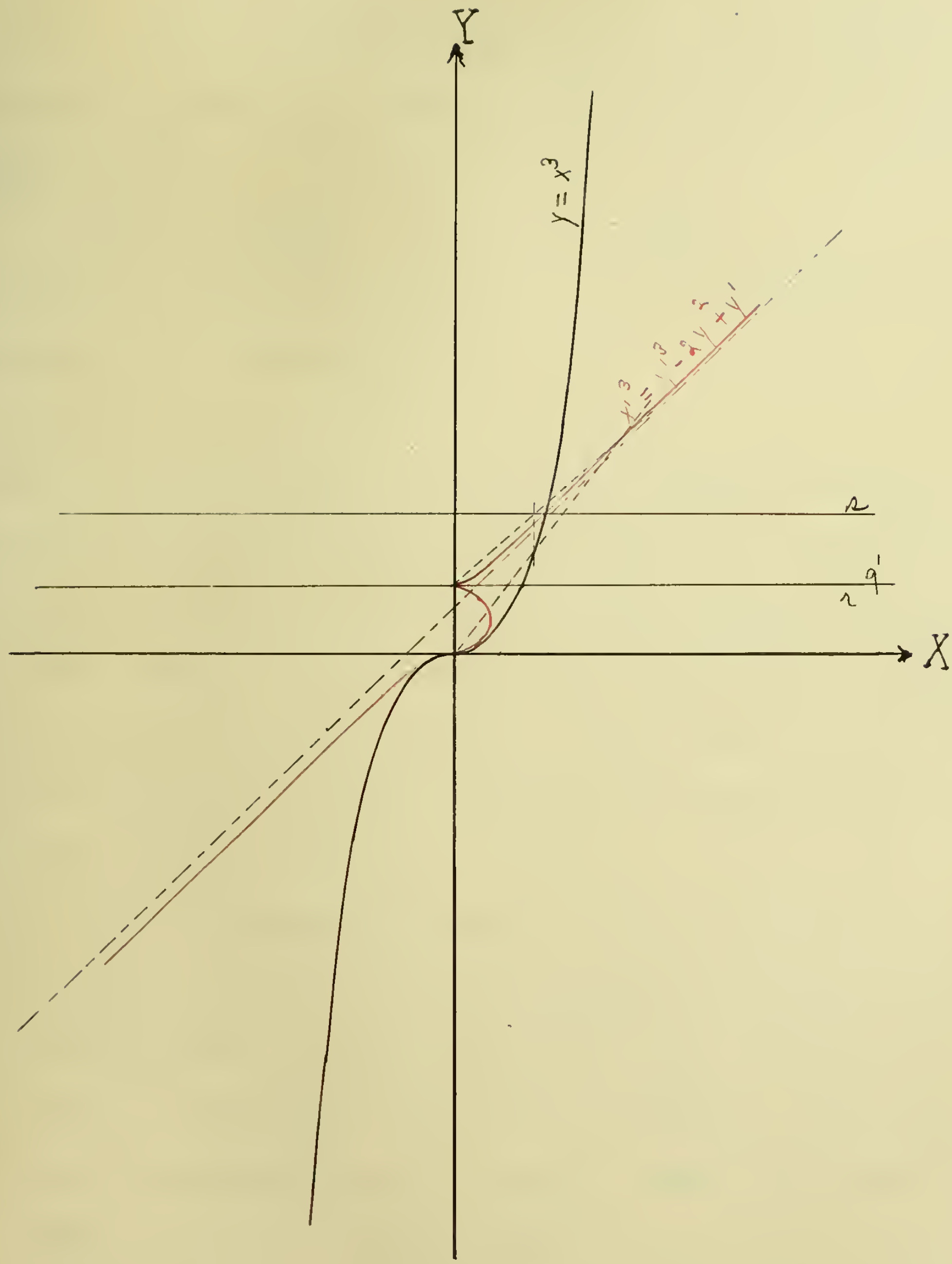

Fig. 5.

CHAPTER III

ALGEBRAIC AND TRANSCENDENTAL CURVES.

8. $y = x^3$. As a particular application of a perspective transformation consider the cubic $y = x^3$ by means of the transformation

$$(1)\quad x' = \frac{x}{y-1}, \qquad y' = \frac{y}{y-1}.$$

To solve for the invariant line, let $x = x'$ and $y = y'$ and after substituting in (1) the equation is

$$(2)\quad y = 2.$$

This line is denoted in Fig. 5 by s. The infinite points of the P' plane are here transformed into the line $y = 1$ denoted by r and found by putting the denominator of the fractions in (1) equal to zero. Now solving (1) for x and y and equating the denominator of their values to zero we have $y' = 1$ as the projection of the infinite points of the P plane. This line is denoted by q' and is coincident with the line r.

The three fundamental lines are now determined and by the ordinary methods of construction in projective geometry the curve $y = x^3$ is transformed into the curve $x'^3 = y'^3 - 2y'^2 + y'$. The results of this construction are shown in Fig. 5. The curve traced in red is the transformed curve. At the point coincident with q' it has a cusp and therefore the original curve has a cusp at infinity. The position of the asymptote to the curve is also shown in the figure.

We now proceed to verify these results analytically. Substituting (1) in the equation of the cubic and simplifying we

have

$$(3)\quad x'^3 = y'^3 - 2y'^2 + y' .$$

Since we desire to know the form of the curve at the point (0,1) we write the equation of a secant through this point:

$$(4)\quad y' = mx' + 1 ,$$

(4) is substituted in (3) and the equation is

$$(5)\quad x'^3 = (mx' + 1)\, m^2x'^2 ,$$

or

$$(6)\quad x'^2\,(m^3x' + m^2 - x') = 0 .$$

From this it is evident that two branches of the curve have $x' = 0$ at the point in question. Moreover $m^3x' + m^2 - x' = 0$, and as x' approaches zero $m^2 = 0$, indicating that the slopes of the two branches are equal and have the value zero at this point. For $y' = mx'$, the equation (2) becomes

$$(7)\quad x'\left[(1-m^3)x'^2 + 2m^2x' - m\right] = 0,$$

which shows that besides the intersection at the origin the line intersects the curve at the points determined by

$$(8)\quad x' = \frac{-m^2 \pm \sqrt{m}}{1 - m^3} .$$

For m equal to a negative number the intersection is imaginary while when m is a positive number there are two real distinct values, indicating that the line intersects the curve in two distinct points. Indeed it can be easily shown by substituting numerical values that the points lie on opposite sides of the tangent to the curve at the singular point and hence we have a cusp of the first kind at the point (0,1). The equation of the asymptote to the curve was found by substituting $y' = mx' + b$ in (3) and equating to zero the coefficients of the two highest

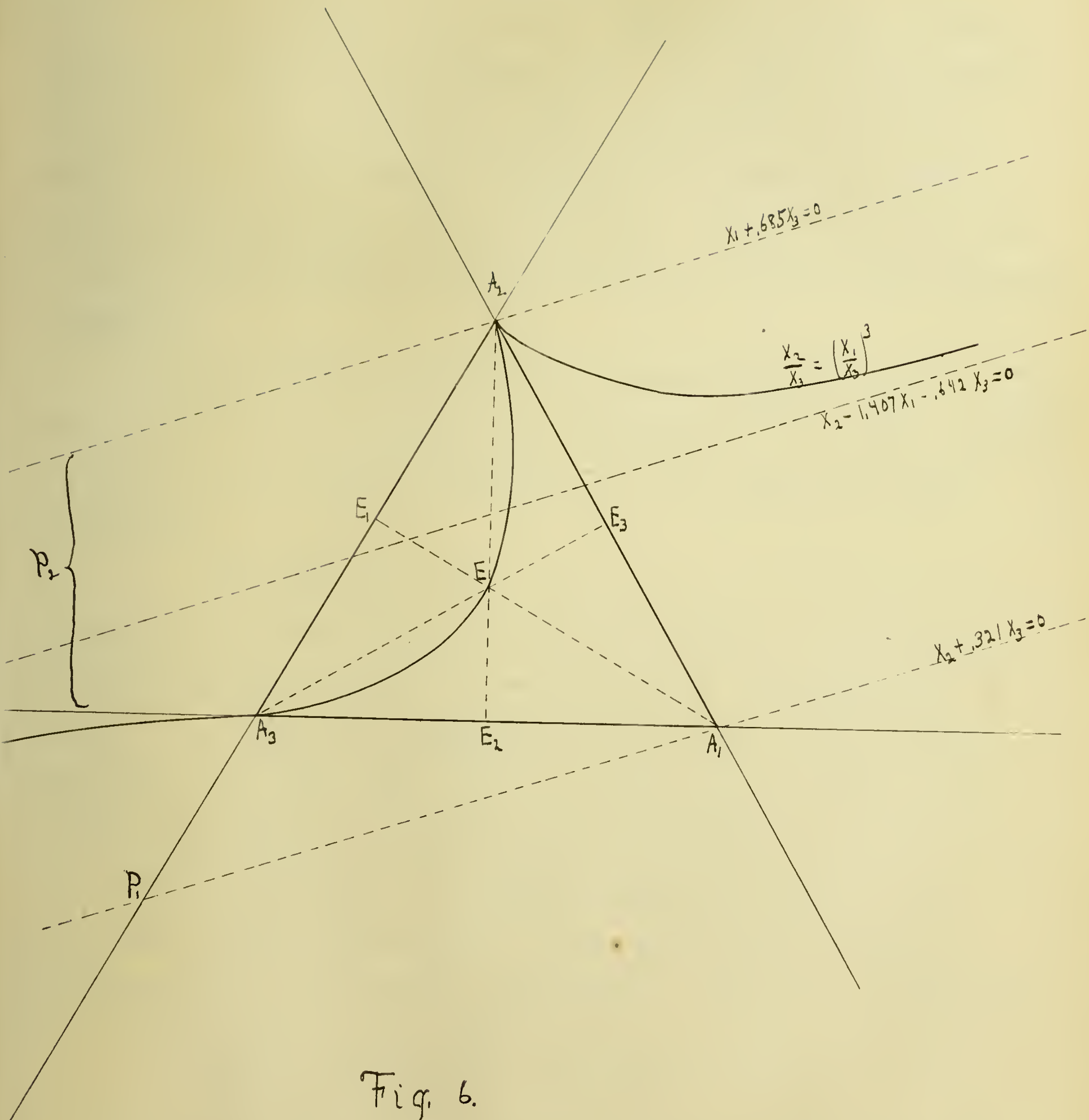
$x_1 + .685x_3 = 0$
A_2
$\frac{x_2}{x_3} = \left(\frac{x_1}{x_3}\right)^3$
$x_2 - 1.407x_1 - .642x_3 = 0$
P_2
E_1
E_3
E
$x_2 + .321x_3 = 0$
A_3
E_2
A_1
P_1

Fig. 6.

powers of the variable. It is given by $3y' = 3x' + 2$. The details of the plotting of the rest of the curve will not be given since this was done in the ordinary way. The study of this curve would suggest that a curve which passes to infinity, on opposite sides of the axis with respect to which it is symmetrical, and in opposite directions has a cusp at infinity. Since we are interested primarily in the character of the curve at infinity the discussion is complete.

We shall now give the details of the discussion of the same cubic by the use of homogeneous coordinates.[(1)] Let $\frac{x_1}{x_3}$ and $\frac{x_2}{x_3}$ be substituted for x and y respectively. As a result the cubic has the form

$$(9)\ x_2 x_3^2 - x_1^3 = 0.$$

The result of plotting the curve is shown in Fig. 6. The curve intersects the line A_1A_2 at the point A_2 whose coordinates are (0,1,0). It is in the neighborhood of this point that we must investigate the curve since A_1A_2 is the line corresponding to the infinite points of the original system of coordinates.

Since this point is a singular point, as in most cases, the tangent becomes indeterminate and we use a method similar to the secant method of Cartesian coordinates. We write the equation of a line through A_2 in the form

$$(10)\ x_1 - \lambda x_3 = 0 .$$

Now solving this with (9) we can divide through by x_3^2 so the curve has a double point at A_2. Moreover the remaining equation is $x_2 - \lambda^3 x_3 = 0$, in which λ becomes infinite as the point of intersection approaches A_2. Therefore the line (1) where $\lambda = \infty$ or indeed

(1) See footnote p. 9.

$x_3 = 0$ is the tangent line to the curve at A_2.

To investigate the character of the curve at the point we solve $x_1 + \lambda x_2 = 0$ simultaneously with (9) with the result that x_3 has two imaginary values. If $x_1 - \lambda x_2 = 0$ is solved with the curve we have $x_3 = \pm\sqrt{\lambda^3 x_2^2}$. This gives a positive and negative value of x_3 and shows that the curve has a cusp at the point A_2.

At the point A_3 the coordinates of which are (0,0,1) the equation of the tangent is $y_2 = 0$. When $x_2 = 0$ is substituted in (9) the result is $x_1^3 = 0$. This shows that the curve is cut in three consecutive points by $x_2 = 0$ and has the line $x_2 = 0$ as the tangent at the point. The curve has a point of inflexion at the point A_3.

We now proceed to obtain the equation of the asymptote to the curve. Assume two lines

$$(11)\quad \begin{cases} x_1 - \lambda x_3 = 0\,, \\ x_2 - \mu x_3 = 0\,, \end{cases}$$

which meet at infinity. Since from these equations $\lambda = \dfrac{x_1}{x_3}$ and $\mu = \dfrac{x_2}{x_3}$, λ and μ are not independent but must satisfy the equation of the curve. Substituting in (9) we obtain the relation

$$(12)\quad \mu = \lambda^3\,.$$

From Fig. 6

$$(13)\quad \frac{x_1}{x_3} = (A_1A_3E_2P_2)\,, \quad \frac{x_2}{x_3} = (A_2A_3E_1P_1)\,,$$

whence

$$(14)\quad \frac{A_1E_2}{A_3E_2} \cdot \frac{A_3P_2}{A_1P_2} = \frac{x_1}{x_3} = \lambda\,,$$

$$(15)\quad \frac{A_2E_1}{A_3E_1} \cdot \frac{A_3P_1}{A_2P_1} = \frac{x_2}{x_3} = \mu\,.$$

But we have

$$(16)\qquad \frac{A_3P_2}{A_3A_1} = \frac{A_3A_2}{A_3P_1},$$

from which

$$(17)\qquad \frac{A_3P_2 - A_3A_1}{A_3P_2} = \frac{A_3A_2 - A_3P_1}{A_3A_2},$$

and

$$(18)\qquad \frac{A_1P_2}{A_3P_2} = \frac{P_1A_2}{A_3A_2} \quad \text{or} \quad \frac{A_3P_2}{A_1P_2} = \frac{A_2A_3}{A_2P_1} .$$

Substituting in (10)

$$(19)\qquad \frac{A_1E_2}{A_3E_2} \cdot \frac{A_3P_2}{A_1P_2} = \frac{A_1E_2}{A_3E_2} \cdot \frac{A_2A_3}{A_2P_1} = \lambda .$$

Now $A_3P_1 = A_2P_1 - A_2A_3$,

and substituting in (15)

$$(20)\qquad \frac{A_2E_1}{A_3E_1} \cdot \frac{A_2P_1 - A_2A_3}{A_2P_1} = \mu = \lambda^3 .$$

From (19) we get,

$$(21)\qquad A_2P_1 = \frac{A_1E_2}{A_3E_2} \cdot \frac{A_2A_3}{\lambda} .$$

Substituting (21) in (19)

$$(22)\qquad \frac{A_2E_1}{A_3E_1} \cdot \frac{\frac{A_1E_2}{A_3E_2}\left(\frac{1}{\lambda} - 1\right)}{\frac{A_1E_2}{A_3E_2} \cdot \frac{1}{\lambda}} = \lambda^3 .$$

Simplifying we have

$$(23)\qquad A_1E_2 \cdot A_3E_1 \cdot \lambda^3 - A_2E_1 \cdot A_3E_2\lambda - A_1E_2 \cdot A_2E_1 = 0 .$$

Since the unit point of the triangle of reference is taken as the intersection of the medians and since the triangle is equilateral the equation reduces to

$$(24)\qquad \lambda^3 + \lambda + 1 = 0 .$$

Solving for λ we have for the real root

$$(25)\qquad \lambda = -.685,$$

which when substituted in (12) gives

$$(26)\ \mu = -.321 \quad .$$

Now when (25) and (26) are substituted in (10) and then y_1 y_2 and y_3 replaced by x_1, x_2 and x_3 respectively the equation of the asymptote is in the form

$$(27)\ x_2 - 1.407\ x_1 - .642x_3 = 0 \quad .$$

When this equation is solved with (9) we get the intersection of the curve with the asymptote which lies in the finite portion of the plane. The coordinates of this point are found to be (1.38,2.628,1). The asymptote then passes through this point and is parallel to the lines (7) which have the form

$$(28)\begin{cases} x_1 + .685x_3 = 0 , \\ x_2 + .321x_3 = 0 , \end{cases}$$

after substituting the values of λ and μ. This completes the discussion by the use of homogeneous coordinates and we find that our results correspond precisely to those obtained by the previous method. Moreover this method of plotting shows the path by which the curve becomes infinite, in a simpler way than the other.

We shall use this method in the discussion of some of the other curves, when it proves advantageous, but the details will not be given as in this case.

9. $2y^2(x-1) = x$. In the discussion of this curve it is found more convenient to take the projective transformation in the form

$$(1)\ x = \frac{x'}{x'-1}, \qquad y = \frac{y'}{x'-1} .$$

By the use of the same method as in the preceding paragraph we find r and q' to coincide and to be the line $x' = 1$. Also s is determined as before and is the line $x = 2$.

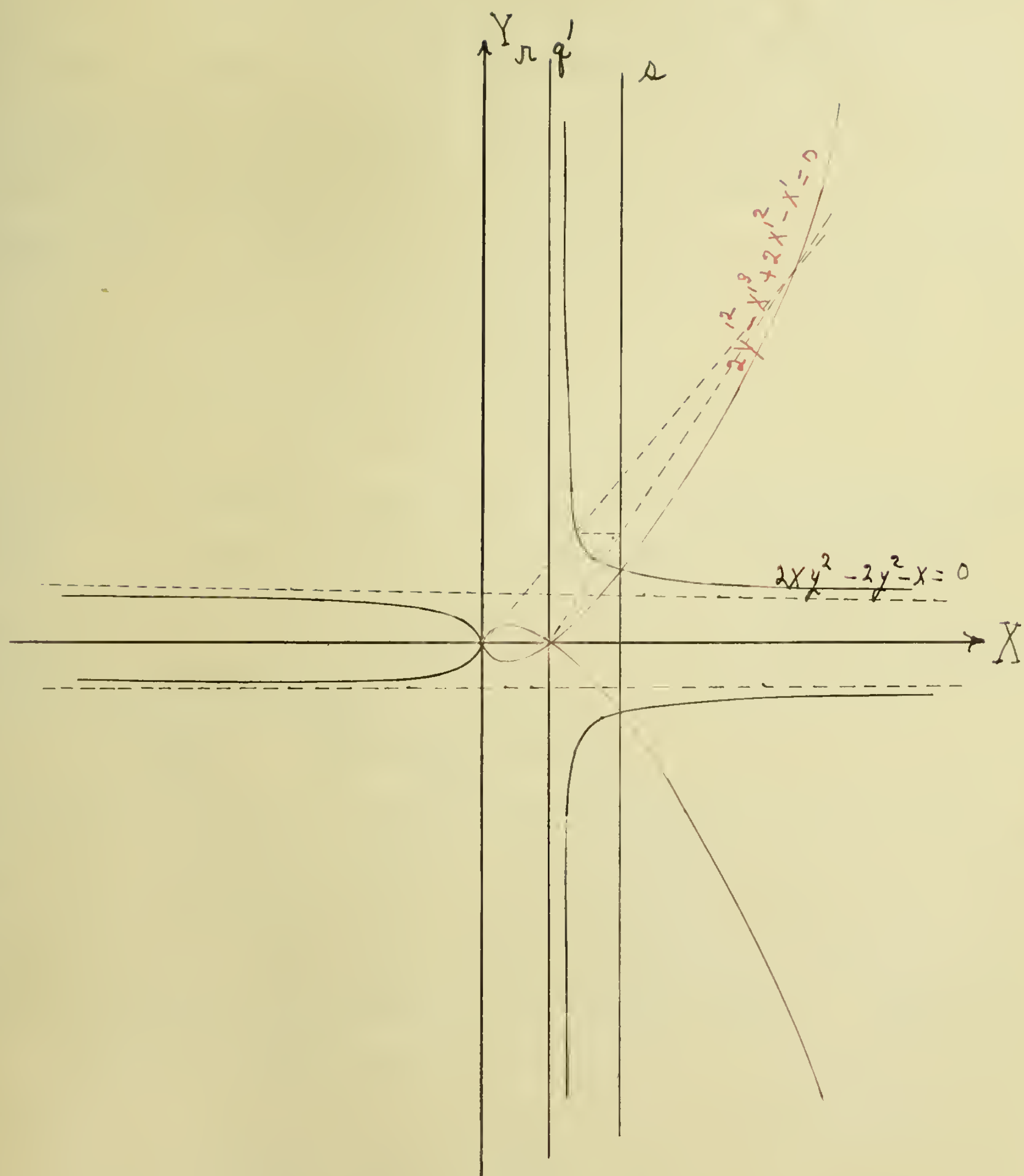

Fig. 7.

The plotting of $2y^2(x-1) = x$ is given in Fig. 7. The curve has three asymptotes and six branches which become infinite. Since four of the infinite branches are symmetrical with respect to the x axis we have chosen the transformation (1) so that the infinite point of these four branches will be transformed to the finite region. As the original curve is obtained by ordinary plotting methods and since we are interested especially in the transformed curve we immediately substitute (1) in the equation of the curve and proceed to investigate the singularities. We obtain by substituting and simplifying.

$$(2)\ 2y'^2 - x'^3 + 2x'^2 - x' = 0.$$

By construction methods we see that the curve has a double point at infinity. We shall establish this property by analytical methods.

By differentiating (2) we have

$$(3)\ \frac{dy'}{dx'} = \frac{3x'^2 - 4x' + 1}{4y'},$$

which is indeterminate at the projected infinite point (1,0). Therefore the equation of a secant through this point is substituted in the equation of the curve and the points of intersection allowed to approach coincidence at the given point. We place

$$(4)\ x' = my' + 1,$$

whence

$$(5)\ y'^2 \left[m^3y' + (m^2 - 2)\right] = 0.$$

Therefore the curve has two branches at the point where $y' = 0$, and since $m^2 - 2 = 0$, or $m = \pm\sqrt{2}$ the slope of these branches is $+\sqrt{2}$ and $-\sqrt{2}$ respectively. Indeed the fact is now established that a curve having two asymptotes which are parallel and which have ordinary tangencies to their asymptotic branches has a double

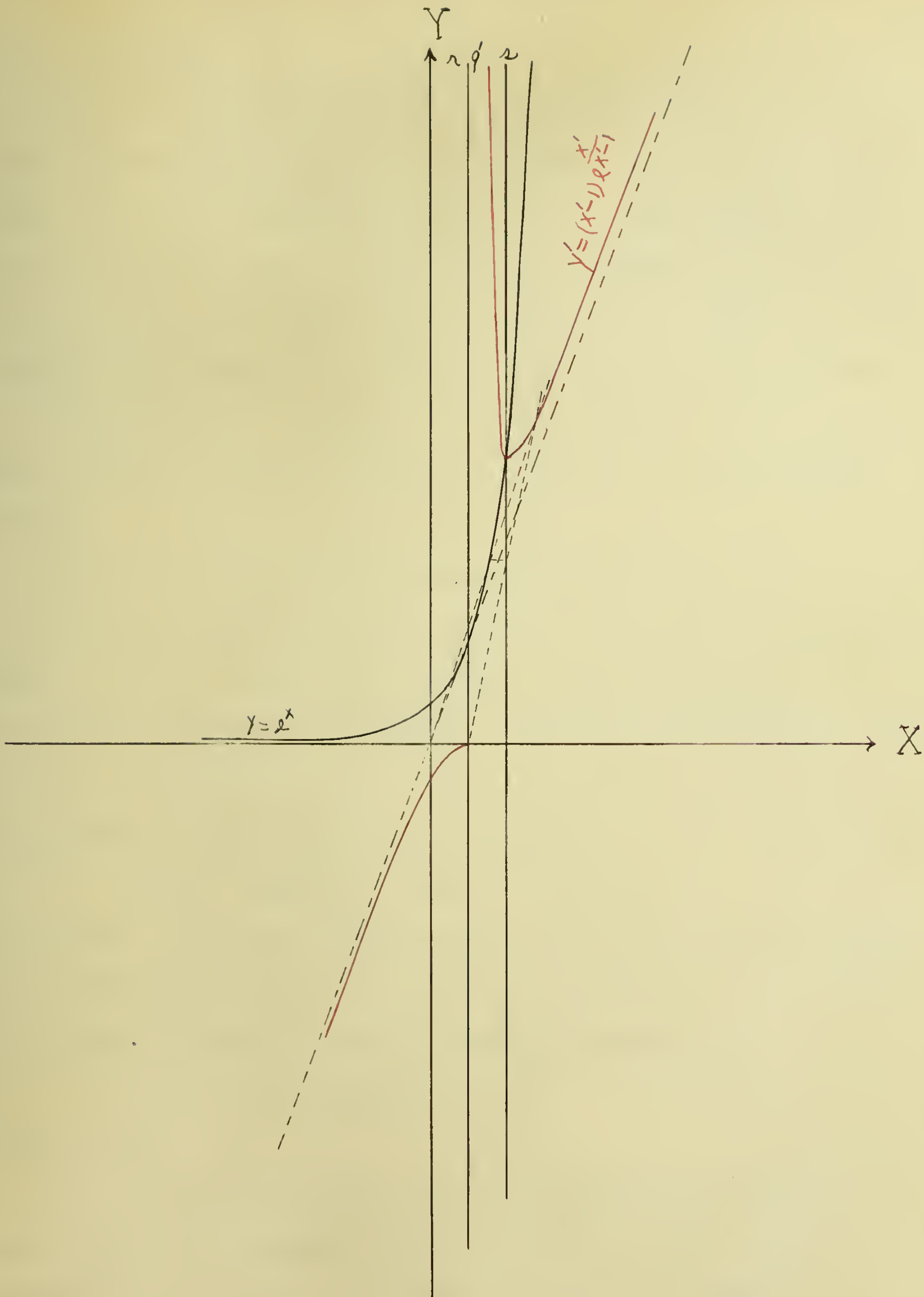

Fig. 8.

point at infinity.

By this transformation of the curve we still have two infinite branches which were infinite before the transformation. We shall now make a statement which will be proven analytically in a later article:- i.e. a curve which proceeds to infinity on the same side of the axis with respect to which it is symmetrical and in opposite directions has a point of inflexion at the infinite point. This statement follows as a result of ∯3. It may also be readily shown by the use of the transformation in the previous paragraph.

10. $y = e^x$ In the discussion of transcendental curves which are given here two important results are manifest. The first is that a curve with a single asymptotic branch has a point of discontinuity at infinity. This is shown when the curve

$$(1)\ y = e^x,$$

is transformed by use of the transformation

$$(2)\ x = \frac{x'}{x' - 1}\ ,\quad y = \frac{y'}{x' - 1}.$$

Here again $x' = 2$ is the invariant line while $x' = 1$ is the line into which the infinite points of (1) are projected. The graphs of the original curve (1) and the projected curve

$$(3)\ y' = (x'-1)e^{\frac{x'}{x'-1}},$$

are shown in Fig. 8.

It is evident from the form of the equation that the negative x axis is an asymptote of the curve $y = e^x$. For if we substitute $x = -\infty$ in this equation and in the value of $\frac{dy}{dx}$ we obtain $y = 0$ and the slope of the curve is also equal to zero. Hence the above result.

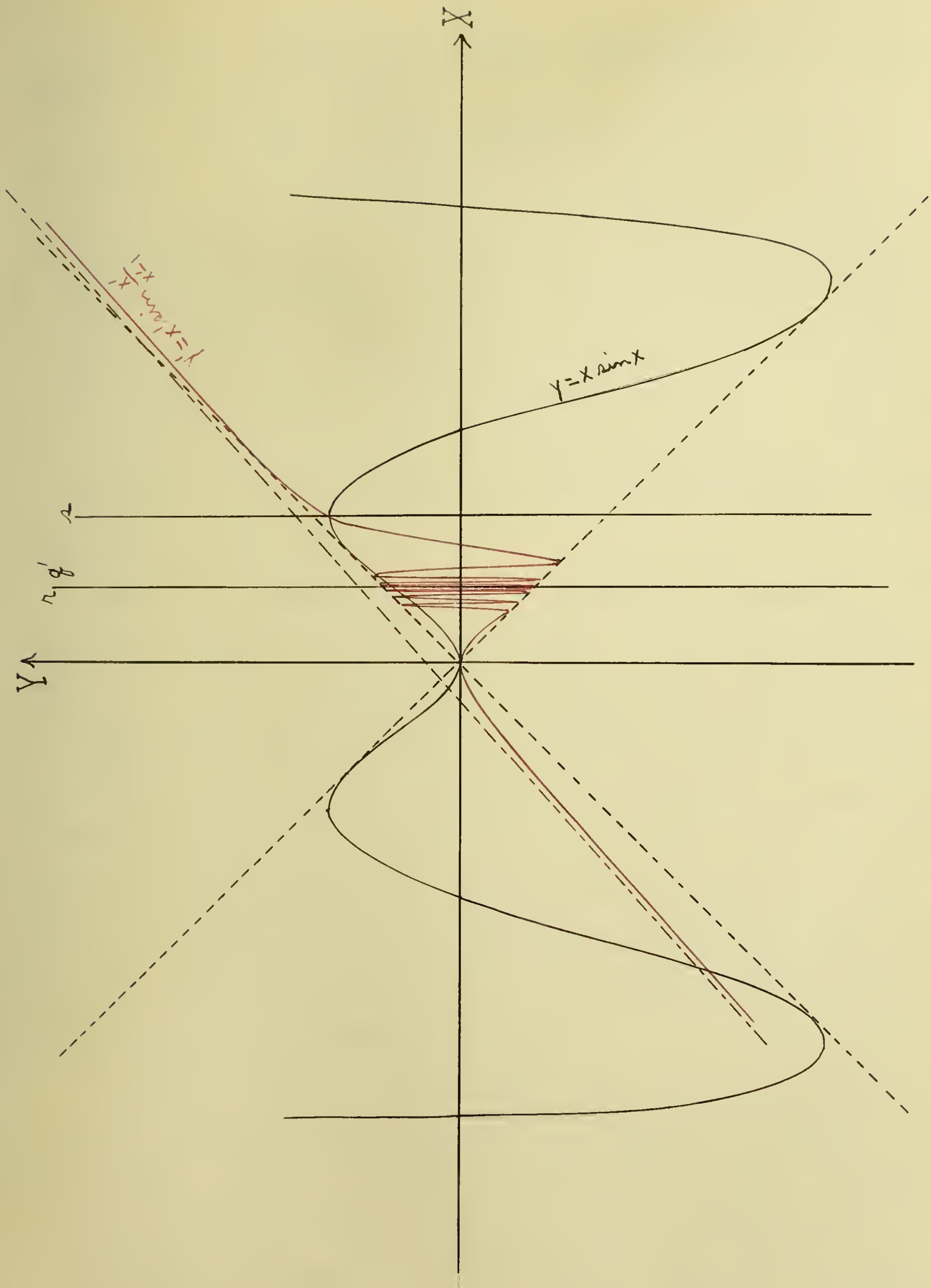

Fig. 9.

Consider the projected curve. Differentiating (3) the slope of the curve is given by

$$(4)\quad \frac{dy'}{dx'} = \frac{x' - 2}{x' - 1}\, e^{\frac{x'}{x' - 1}}.$$

From (3) it is clear that for $x' = 1$ the value of y' becomes indeterminate and therefore we must have a point of discontinuity. As this line is approached from the left hand side the values of (3) and (4) approach zero and hence there is a branch ending at (1,0) which is tangent to the x axis at the point. When the line $x' = 1$ is approached from the right hand side the values of (3) and (4) become infinite and a branch of the curve is then asymptotic to this line.

The other asymptote of (3) is determined graphically by drawing the tangent to the original curve at the point where it crosses $x' = 1$, then projecting any two convenient points of the tangent and drawing a straight line through these points gives the asymptote to the curve. To obtain the asymptote analytically we write the equation of the tangent at the point where the curve crosses r; then projecting the line in the same manner as the curve was projected we have the asymptote to the transformed curve. It is clear that the result stated in the first of this article follows as a result of this discussion.

11. $y = x \sin x$. It is easily seen that this curve is periodic and as x becomes infinite the value of y becomes indeterminate. It is of interest to note that the curve passes through the origin and that y returns to the value zero for x equal to the multiple values of $\frac{\pi}{2}$ either positive or negative. The form of the curve is shown in Fig. 9.

Substituting

$$(1)\ x = \frac{x'}{x'-1}, \quad y = \frac{y'}{x'-1},$$

in $$(2)\ y = x \sin x,$$

the resulting equation is

$$(3)\ y' = x' \sin \frac{x'}{x'-1}.$$

It is clear that in the neighborhood of $x' = 1$ the oscillations of the curve become dense and at $x' = 1$ are indeterminate. We have then the second important result from the discussion of transcendental curves:- i.e. the oscillations of the function (2) which is not defined at infinity become dense at infinity. This fact is often made use of in the discussion of certain theorems in the theory of functions.

From (2) by differentiating there results

$$(4)\ \frac{dy}{dx} = x \cos x + \sin x.$$

Substituting $x = 1$ in (2) and (4) the equation of the tangent at the point where (2) crosses r is

$$(5)\ y - 1.3819x = -.5405,$$

which when transformed gives

$$(6)\ y' - .8414x' = .5405,$$

as the equation of the asymptote to the projected curve.

From (2) it is seen that the maximum value of y is equal to the maximum value of x. Therefore the curve oscillates between the lines $x + y = 0$ and $x - y = 0$. Moreover the portion of the curve lying to the left of the y axis is projected into the region between the y axis and $x' = 1$ and vice versa. Besides the portion of the curve to the right of the line $x' = 2$ is transformed into the region lying between $x' = 1$ and $x' = 2$ and vice versa.

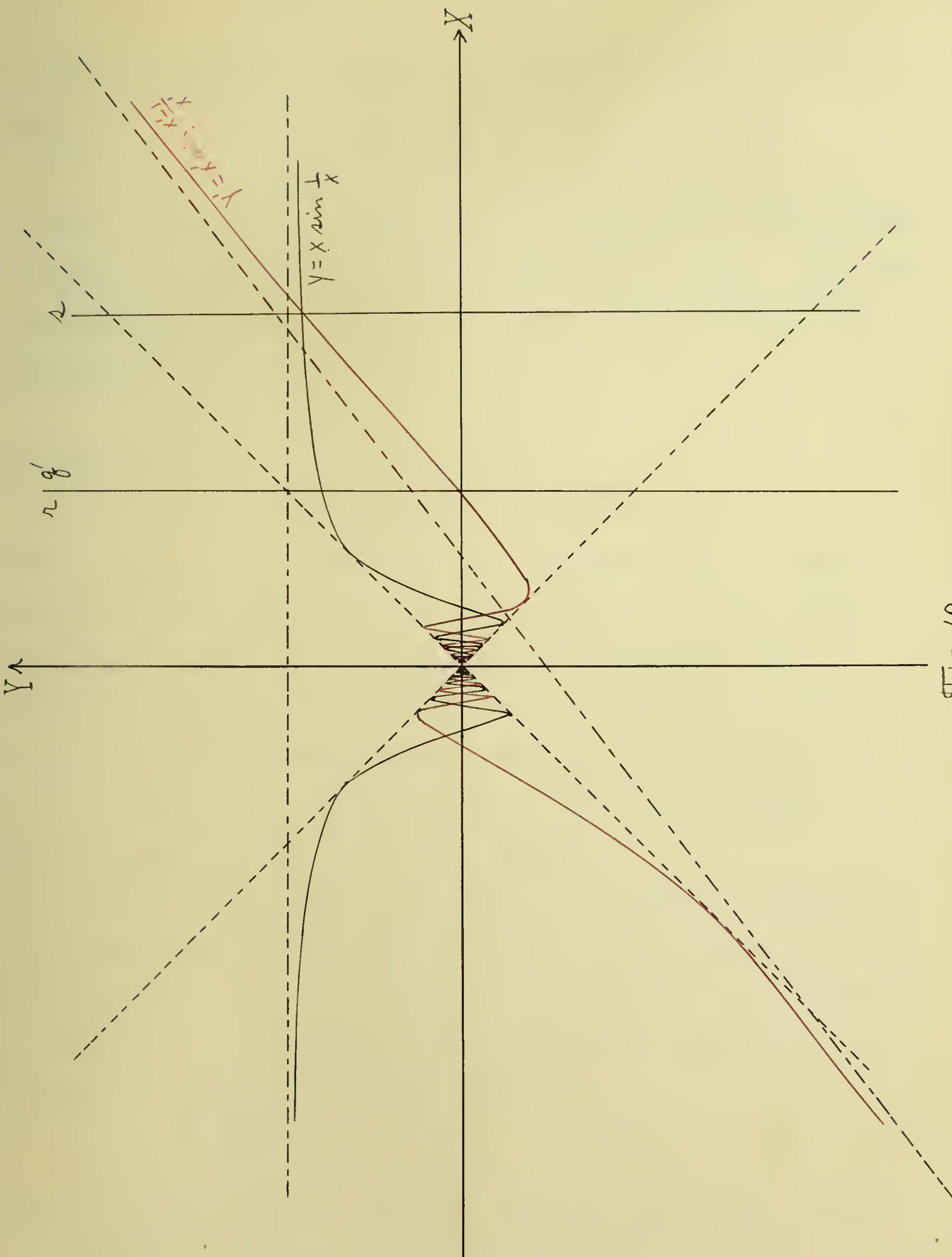

Fig. 10.

This fact is often times useful in constructing certain complicated curves.

12. $y = x \sin \frac{1}{x}$. The discussion of this curve does not bring forth any new properties of a perspective transformation but merely verifies some of those already established. The same transformation is used as in the previous article and when substituted in

$$(1)\quad y = x \sin \frac{1}{x},$$

the resulting curve is represented by

$$(2)\quad y' = x' \sin \frac{x'-1}{x'}.$$

From the graph of (1) in Fig. 10 it is evident that the curve oscillates between the lines $x + y = 0$ and $x - y = 0$ and these oscillations become dense at the origin. Besides as the values of x become large y approaches the value 1 as a limit and the value of the derivative approaches zero. Hence the curve has $y = 1$ as an asymptote and passes off to infinity in opposite directions on the same side of the asymptote.

By examining (2) we can readily see that the transformed curve oscillates between the same lines as the original curve and its oscillations also become dense at the origin. To determine the character of the singular point at infinity equation (2) is differentiated twice, the result being

$$(3)\quad \frac{d^2y'}{dx'^2} = \frac{-1}{x'^3} \sin \frac{x'-1}{x'}.$$

Where $x' = 1$, the value of $\frac{d^2y'}{dx'^2} = 0$ and the curve has a point of inflexion at infinity. This accords with the result obtained in article 4. Proceeding as in the previous article to

find the asymptote we have

$$(4)\ y'-.8414x' = -.5405,$$

as the desired equation.

From the discussion of the last three curves a method for determining the asymptotes to a curve suggests itself, which might be useful when the curves under disucssion are transcendental. The method consists in projecting the infinite point of the curve to a finite point, then writing the equation of the tangent to the projected curve at this point. The transformation of this line to the original system of coordinates will be the asymptote to the given curve.

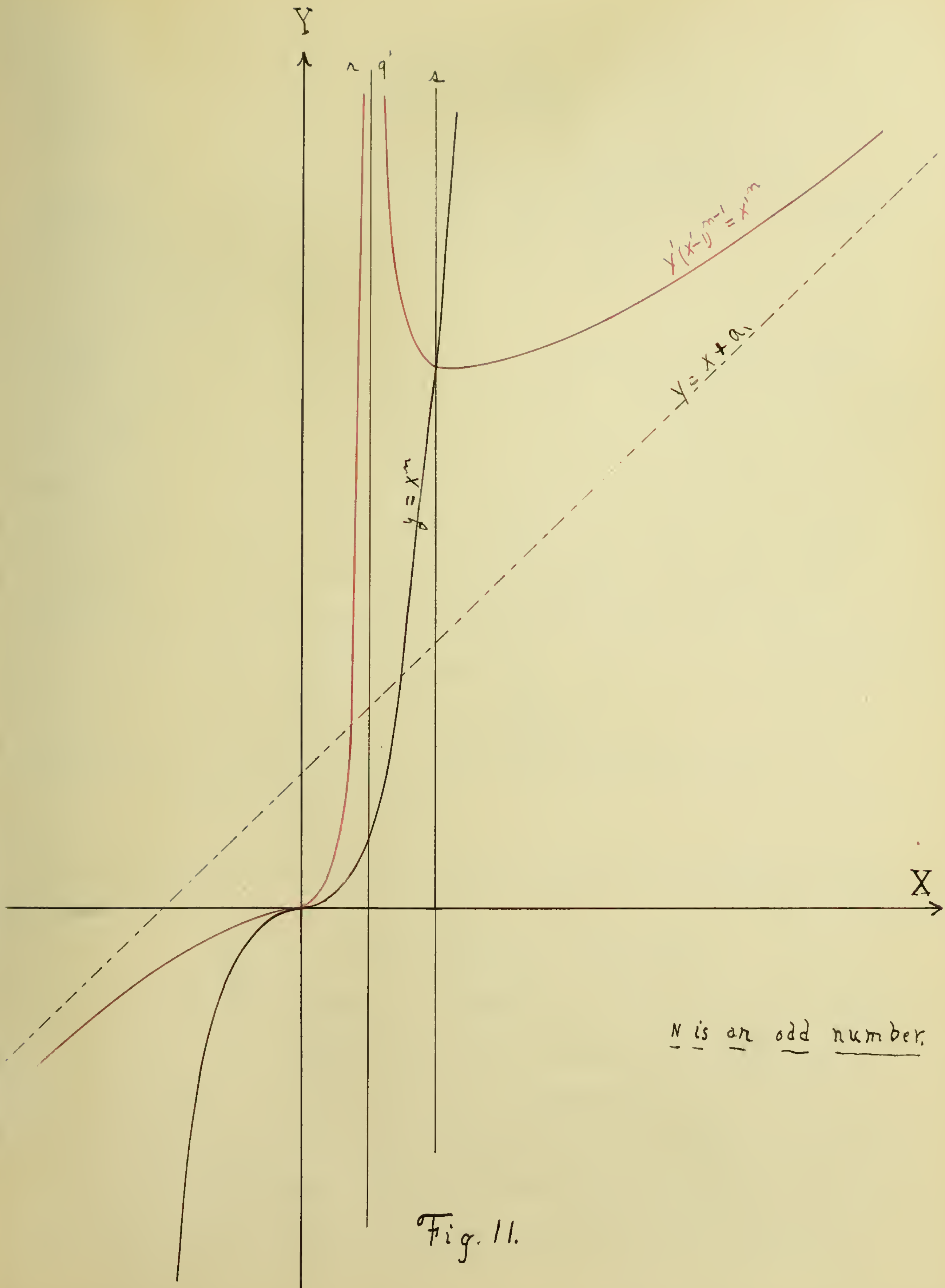

Fig. 11.

CHAPTER IV.

BINOMIAL CURVES.

13. $y = ax^n$. We now proceed to investigate the character at infinity of a set of curves called binomial curves. The general equation of such a set of curves is

$$(1)\quad y = ax^n,$$

where a is a constant and for convenience may be assumed equal to one, while n is any positive constant. Two general curves arise from this equation, the one when n is an even number and the other when n is an odd number. Indeed it is at once evident from the form of the equation that the branches of the curve for which x is positive would be similar and the branches for which x is negative would merely change quadrants for these two cases.

Here again the transformation

$$(2)\quad x = \frac{x'}{x'-1}, \qquad y = \frac{y'}{x'-1},$$

is used and by projecting (1) the resulting equation is

$$(3)\quad y' = \frac{x'^n}{(x'-1)^{n-1}}.$$

Differentiating (3) the slope of the curve at any point is found to be

$$(4)\quad \frac{dy'}{dx'} = \frac{nx'^{(n-1)}(x'-1) - (n-1)x'^n}{(x'-1)^n}.$$

Now when n is an odd number the plotting of the curves is shown in Fig. 11. The original curve has the form of an ordinary cubic and when its infinite point is projected to the line q' it still remains at infinity. Consequently, as is evident from the figure, the transformed curve has two branches asymptotic to the line $x' = 1$, and coming in the same direction from infinity on

[illegible]

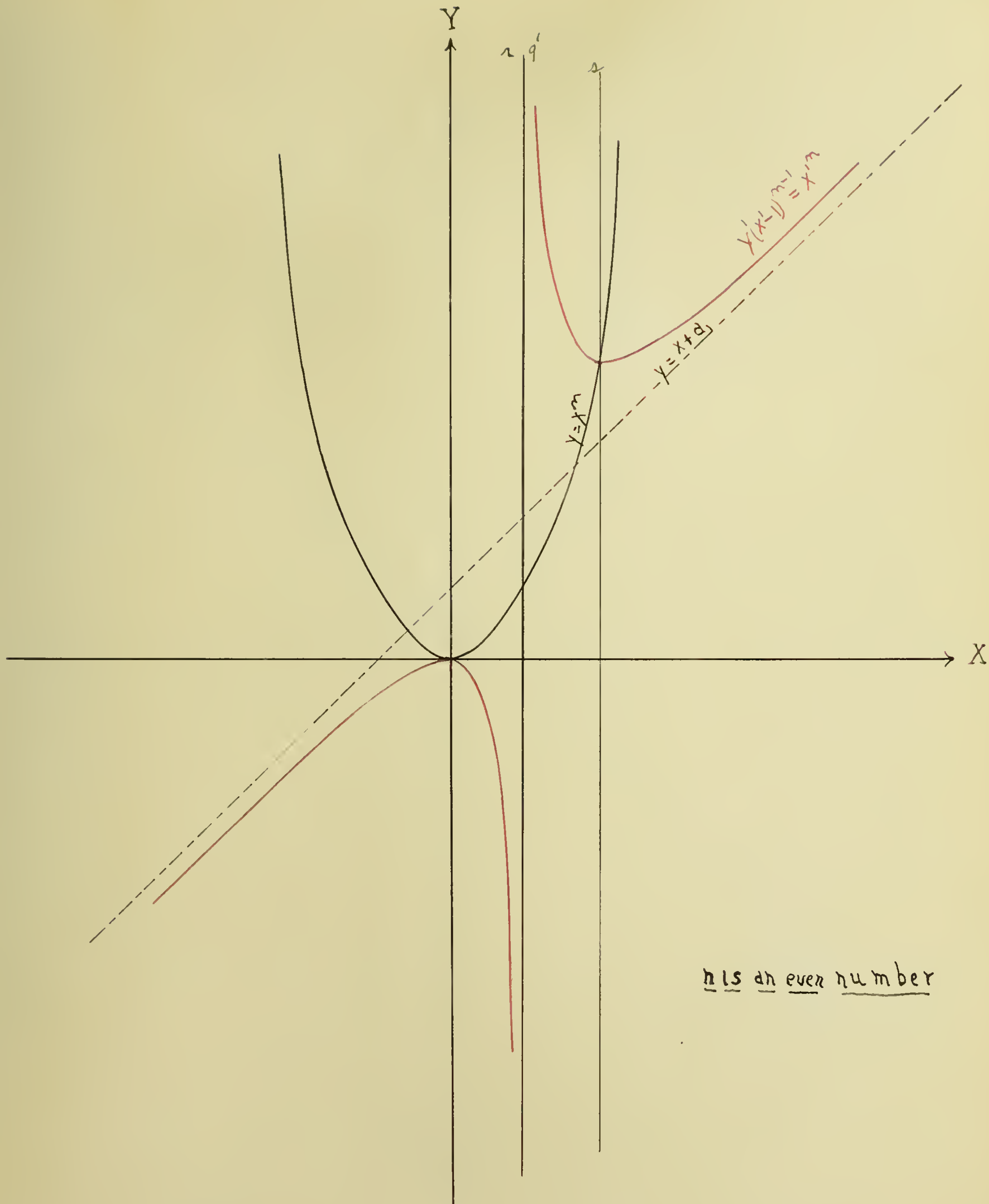

Fig. 12.

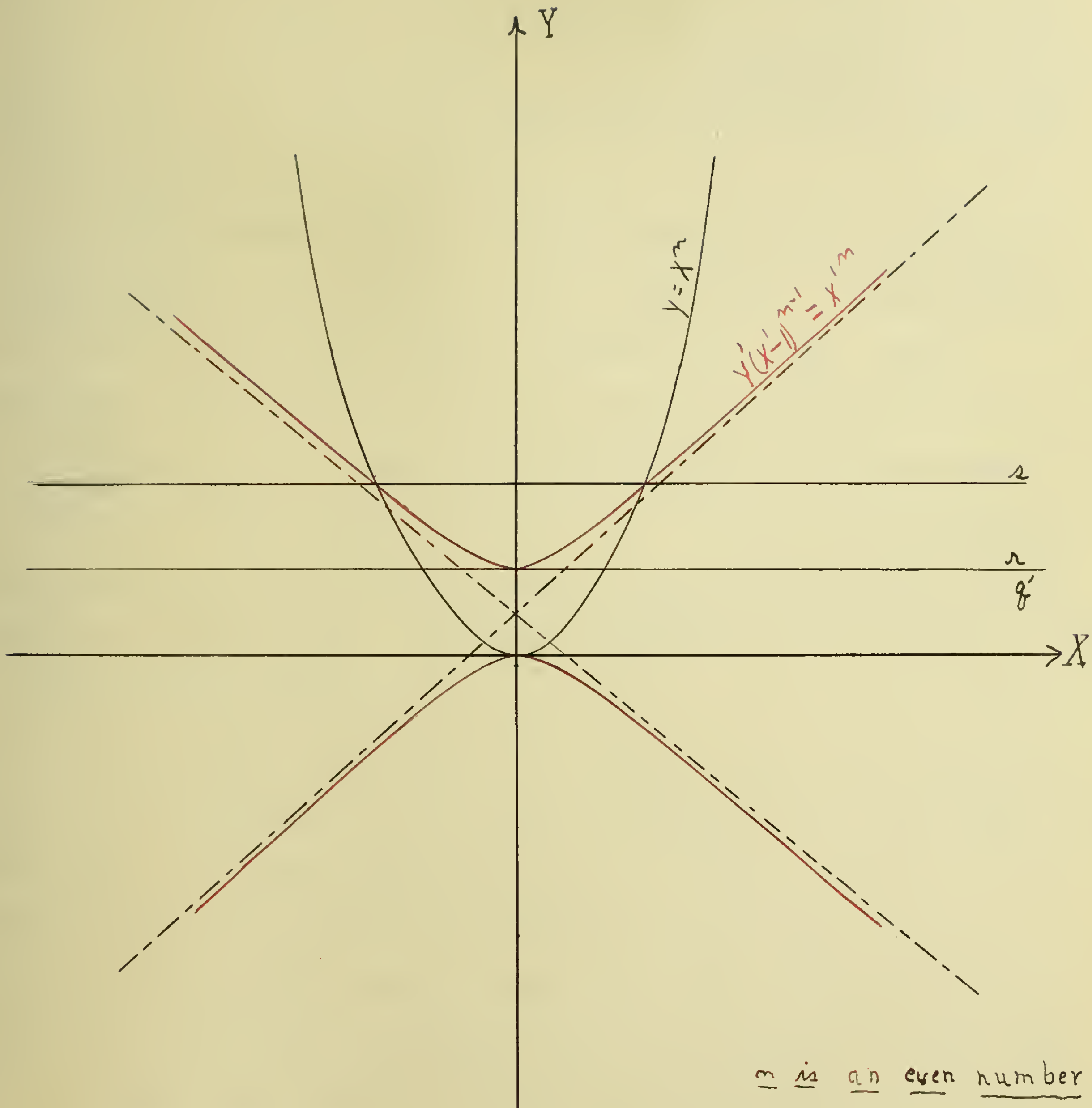

Fig. 13.

opposite sides of the asymptote. Interpreting this result by article 4 the curve has as a first approximation a cusp at infinity in the direction of the y axis. If the transformation of article 7 had been used the transformed curve would have had the same form as the projected curve in Fig. 5.

These results may be verified analytically as follows:- For the regions $0<x'<1$ and $1<x'$, y' is positive. Substituting $x' = 1$ in (3) and (4) y' equals infinity, and $\frac{dy'}{dx'}$ equals infinity. Therefore as stated above the curve is asymptotic to the line $x' = 1$. Again these results must be interpreted by the use of the results of article 4. Or if the transformation of article 7 is used the results may be interpreted by the method of that article. The other asymptote to the curve is found by the ordinary method to be $y = x + a_1$.

If n is an even number the original curve is of parabolic form with the vertex at the origin and symmetrical with respect to the y axis. For the region $0<x'<1$, y' is negative while for the region $1<x'$, y' is positive. The result of substituting $x' = 1$ in (3) and (4) is the same as when n was odd. From these facts it is clear that the transformed curve is asymptotic to the line $x' = 1$, comes from infinity in opposite directions and on opposite sides of the asymptote. Again from the results of article 4 it is seen that the curve has at a point at infinity the line at infinity as a tangent. The other asymptote to this curve is the same as the one above except that the a_1 is not necessarily the same constant since it depends on the coefficients of the powers of x'. The plotting for this curve is shown in Fig. 12. In Fig. 13

the result of the use of the transformation in article 7 is shown. The above results are verified by this transformation.

14. $y = ax^{p/q}$. Another important class of the binomial curves $y = ax^n$ is found when n is a proper fraction and p and q are primed to each other. Without loss of generality, a is assumed equal to one. These curves will be discussed first from the standpoint of the theory of higher plane curves after which a brief topological discussion will be given.

Let

$$(1)\quad y = x^{p/q}$$

be changed to homogeneous coordinates; then after simplifying it assumes the form

$$(2)\quad x_2^q = x_1^p x_3^{q-p},$$

which when divided by x_1^q is

$$(3)\quad \left(\frac{x_3}{x_1}\right)^{q-p} = \left(\frac{x_2}{x_1}\right)^q.$$ [1]

For the singularities at the origin of a curve of the form

$$(4)\quad y^m = x^p, \qquad (\text{ p greater than m})$$

we have

$$(5)\quad \delta + \kappa = \frac{(p-1)(m-1)}{2},$$

where δ is the number of double points and κ is the number of cusps. Moreover it is also found that $\kappa = m-1$ at the origin, hence when this value is subtracted from the right hand member of equation (4) the number of double points is determined. It is evident now from equation (3) that when $x_2 = 0$, $x_3 = 0$ and hence the coordinates of the point A_1 satisfy the equation of the curve. Moreover the point A_1 in this case correponds to the origin in equation (4) and since

(1) For a discussion of curves on this form see C. A. Scott-- On the Higher Singularities of Plane Curves. Amer. Jour. Math. Vol. XIV, pp. 322 et seq.

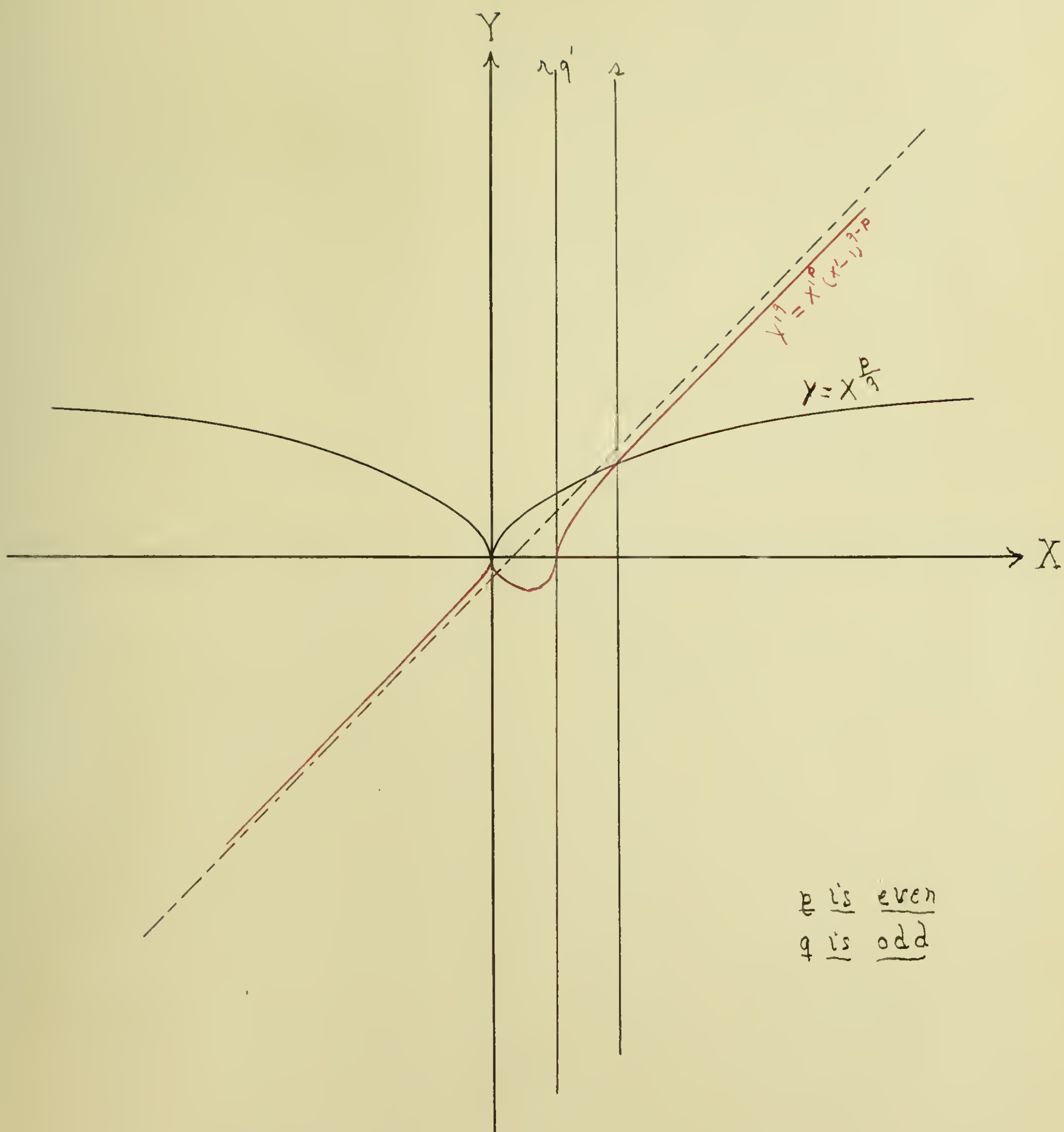

Fig. 14.

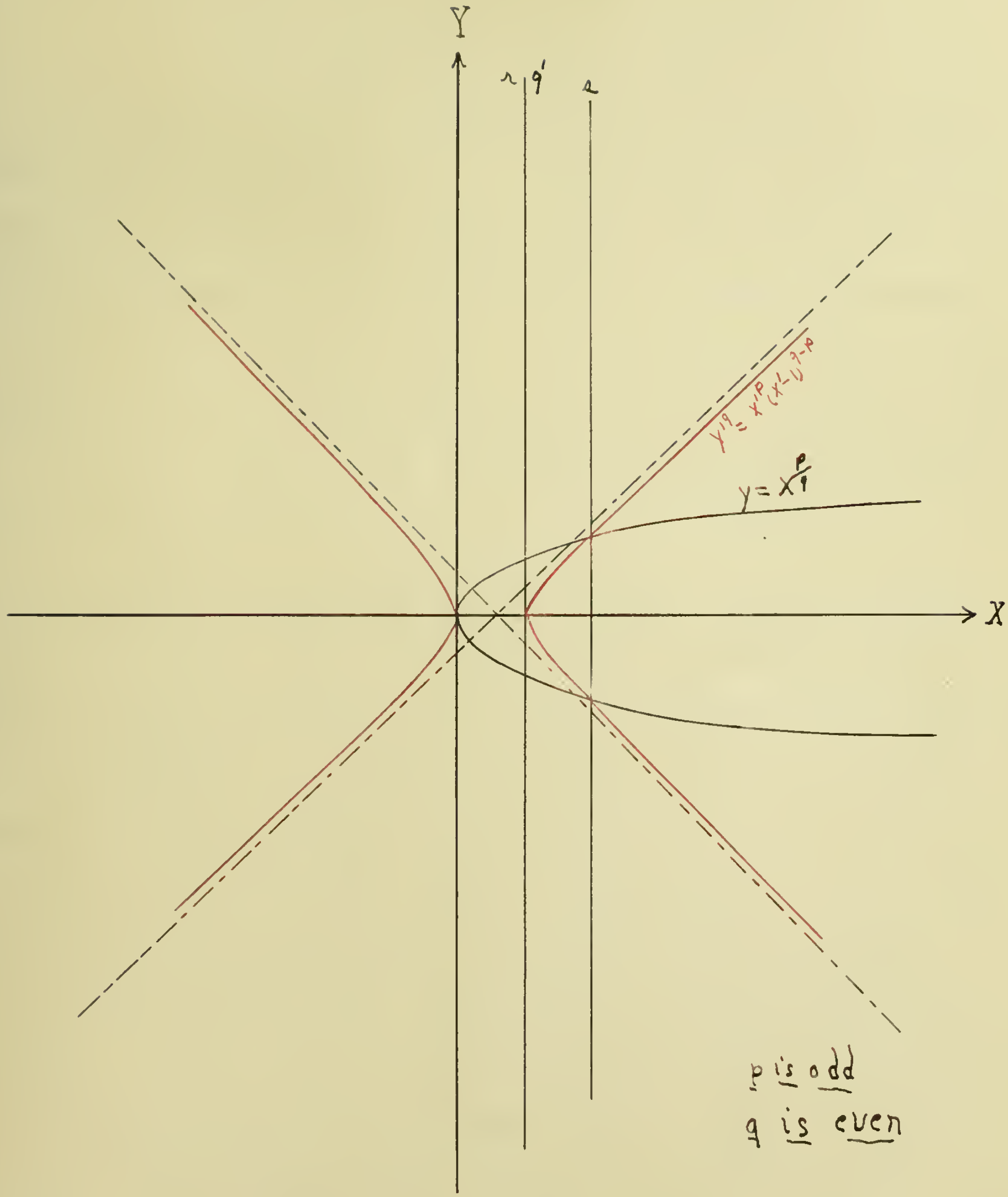

Fig. 15.

A_1 is the projection of the infinite point in the direction of the x axis the curve (2) has at infinity q-p-1 cusps and the number of double points is given by

$$\delta = \frac{(q-1)(q-p-1)}{2} - (q-p-1).$$

Indeed the coordinates of the point A_2 do not satisfy (2) and hence the curve has no infinite point in the direction of the y axis. In case p/q was an improper fraction we would divide (2) by X_2 raised to its highest power and then the point A_2 would correspond to the origin of the equation (4). The number and kind of singularities which occur would be the same as in the discussion just above. A special case of this is given in the previous article where q is taken equal to one.

In the topological discussion of these curves, three distinct cases arise from the values which p and q may take:- i.e. we may have p an even number and q and odd number, p and odd number and q an even number, or finally both may be odd. The details of the results of this study will not be given since the method used is precisely the same as that used in the previous articles and moreover many of the same results have been obtained before. From Fig. 14 we see that we have a semi-cubical parabola symmetrical with respect to the x axis. Clearly from the form of the projected curve we have the cusps at the origin and the points of inflection at infinity. This verifies the statement made in article 8 that a curve which comes from infinity on the same side of the axis with respect to which it is symmetrical and in opposite directions has a point of inflection at the infinite point.

For the second case as shown in Fig. 15 the curve has a

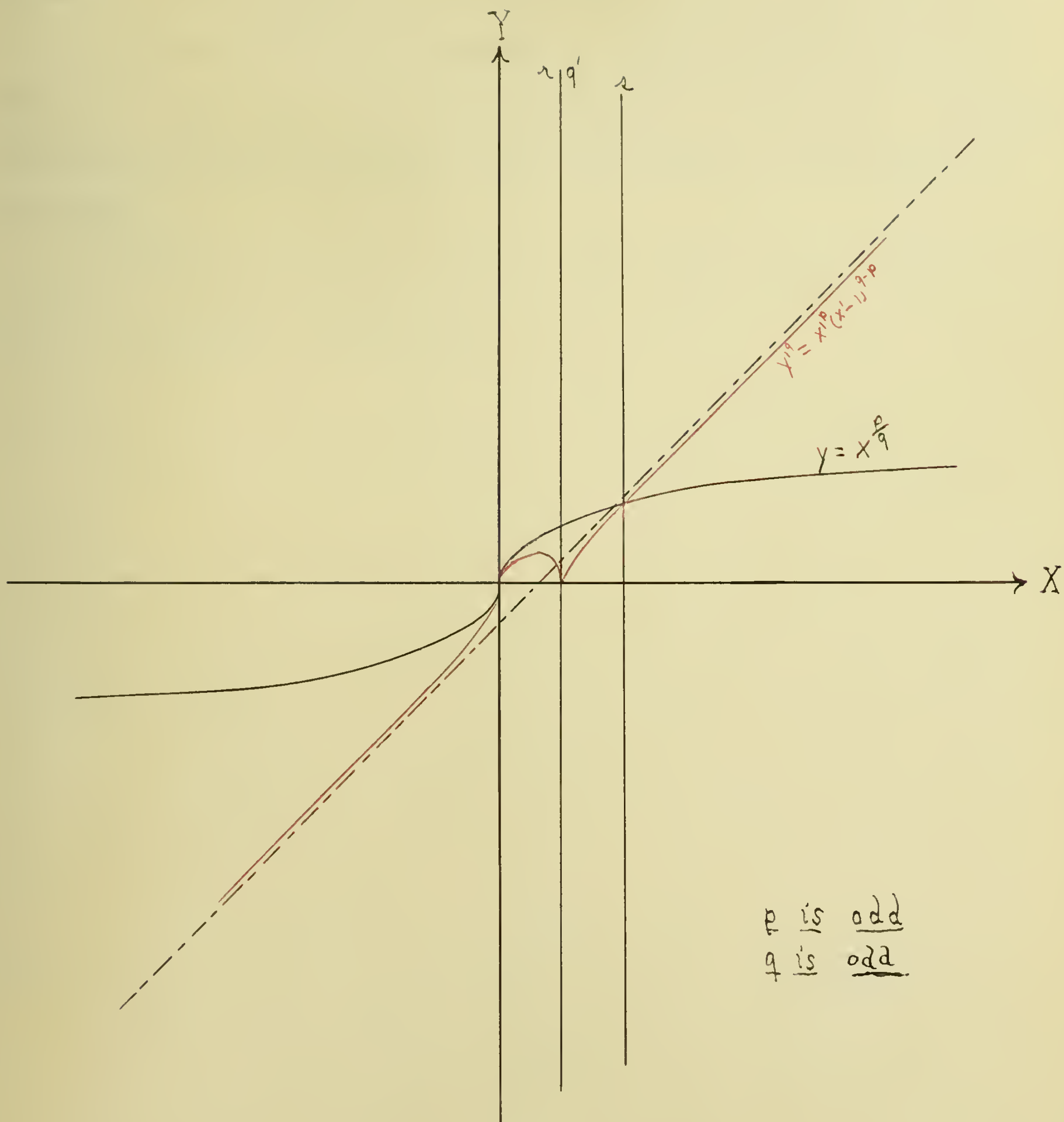

Fig. 16.

parabolic form and consequently has an ordinary tangency at infinity. It is seen that this curve in the neighborhood of the cusps and double points appears as an ordinary branch of the curve.

Finally in the third case Fig. 16 shows that we have a curve of a form which was studied in detail in article 8. Clearly the cusps now occur at infinity and the points of inflection at the origin.

The verification of these results may be easily made by the use of homogeneous coordinates.

CHAPTER V

CYCLIC CURVES.

15. $(x^2 + y^2)^k \phi_{n-2k}(x,y) + \psi_{n-p}(x,y) = 0$. In the preceeding chapters our study has been limited to the form at infinity of those curves which have real branches in that portion of the plane. We shall now extend our study to a class of curves called cyclic curves which may be closed in the finite portions of the plane, but which have imaginary branches passing through the circular points. A general example will be cited, the method of proceedure and interpretation of the results illustrated.

From the definition of a cylic curve it may be taken in the form

$$(1)\ (x^2 + y^2)^k \phi_{n-2k}(x,y) + \psi_{n-p}(x,y) = 0,$$

where 2k is less than n, ϕ is of a degree n-2k and ψ of degree n-p where p equals 1,2,3,-------n. A special case of such curves has already been studied in Chapter II and the method employed there may easily be extended to the general case. For the perspective transformation

$$(2)\ x = \frac{x'}{x'-1},\ y = \frac{y'}{x'-1}.$$

When (2) is substituted in (1) and the result simplified

$$(3)\ (x'^2 + y'^2)^k \phi_{n-2k}(x',y') + (x'-1)^p \psi_{n-p}(x',y') = 0.$$

From (2) it is evident that as x and y become infinite, x' must equal one. Also we have from the same relation that

$$(4)\ y/x = y'/x'.$$

Since x' = 1 substituted in (3) gives the relation

$$(5)\ (x'^2 + y'^2)^k = 0,$$

whence $\quad (6)\ (x' + iy')^k (x' - iy')^k = 0,$

and (7) $\frac{y'}{x'} = \pm i$;

then it is clear that

$$(8) \lim_{\substack{x \doteq \infty \\ y \doteq \infty}} \left(\frac{y}{x}\right) = \pm i.$$

Indeed, the curve (1) may pass k times through each of the two conjugate imaginary points I'_1 and I'_2 and moreover these points are transformed into the conjugate imaginary points of the line $x' = 1$. The problem reduces then to the investigation of the transformed curve at the latter points. To do this the equation

$$(9) \quad y' = m(x'-1) \pm i$$

is solved with (3) and by reduction the number of values of m at the points $(1,i)$ and $(1, -i)$ is found. Each value of m gives the direction of a branch of the curve and hence the multiplicity of the points.

It is of interest to note that when (3) is differentiated the value of dy'/dx' is infinite when p is equal to one so that the curve touches the straight $x'-1 = 0$ at I'_1 and I'_2. For all other values of p dy'/dx' is indeterminate and hence the curve has a multiple point at infinity and its slope is determined by the value of m given above. For $p = 1$ then the branches of the curve, however many they may be, have the line $x' = 1$ as a tangent. Clearly then the original curve passing through the conjugate imaginary points, either have multiple points with distinct or partly coincident tangents, or if p equals 1 the curve touches the infinite line at I_1 and I_2.

The complete construction of the curves will be given but the analytical methods by which the form of the curves at points other than the infinite points will only be indicated, since they are

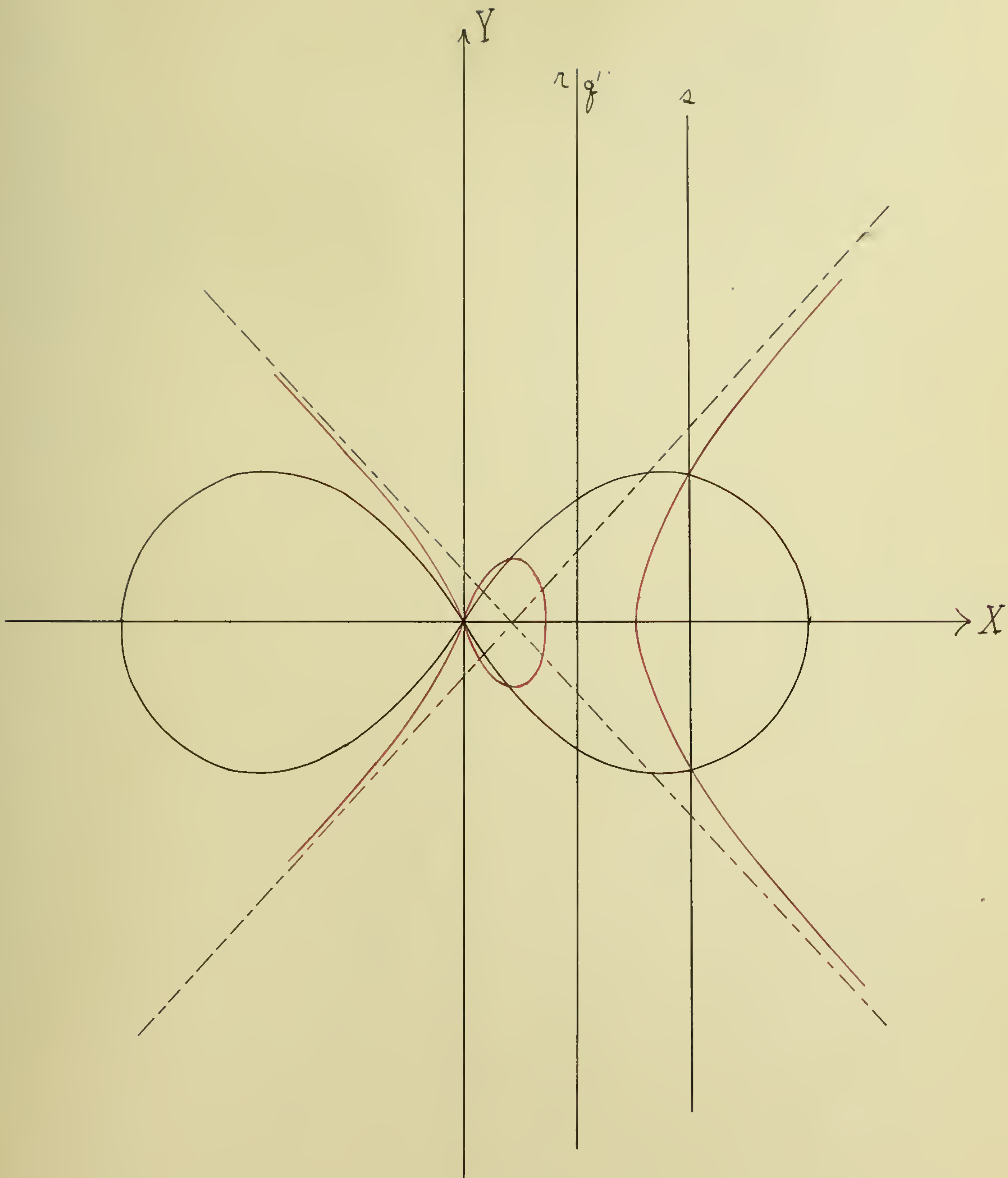

Fig. 17.

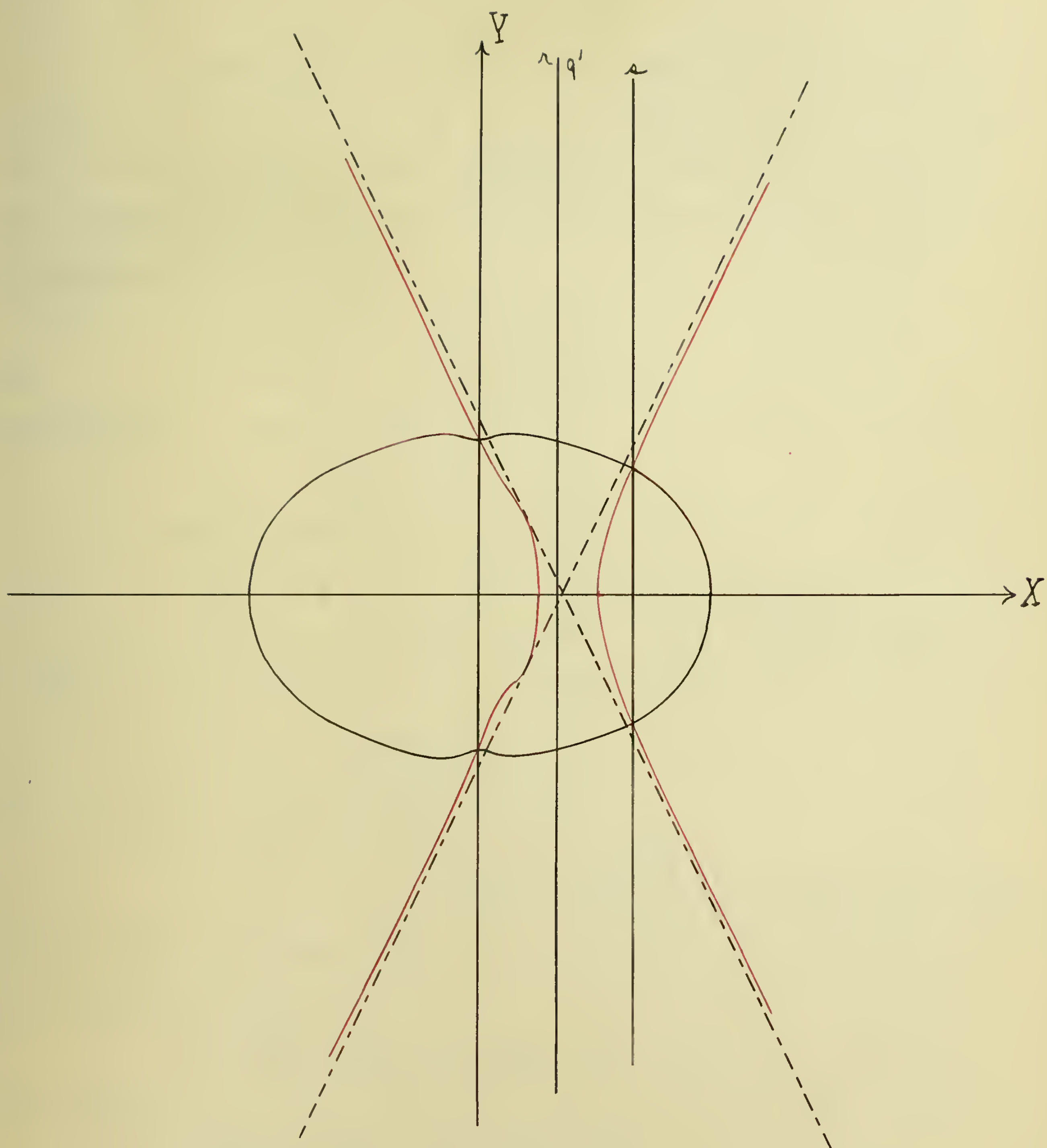

Fig. 18.

determined by ordinary methods and are not of special interest in our study.

16. Lemniscate. As a particular example of the curves just discussed we have selected one which was studied by Jacob Bernoulli and to which he gave the name lemniscate. There are two types of these curves, the one hyperbolic and the other parabolic. They are represented by the equations

$$(1)\ (x^2+y^2)^2 - a^2x^2 + b^2y^2 = 0,$$

and

$$(2)\ (x^2+y^2)^2 - a^2x^2 - b^2y^2 = 0,$$

respectively. Substituting

$$(3)\ x = \frac{x'}{x'-1},\ y = \frac{y'}{x'-1},$$

in (1) the result simplifies to the form

$$(4)\ (x'^2 + y'^2)^2 - (x'-1)^2(a^2x'^2 - b^2y^2) = 0.$$

It is clear from the form of the curves in Fig. 17 and 18 that there are no real points at infinity. Let us substitute in (4)

$$(5)\ y' = m(x'-1) \pm i.$$

When this result is simplified we have

$$(6)\ \left[x' + 1 + m^2(x' - 1) \pm 2im\right]^2 - \left[a^2x'^2 + b^2\left\{m^2(x'-1)^2 \pm 2im(x'-1)-1\right\}\right] = 0.$$

In the limit as x' approaches one the equation is a quadratic in m and of the form

$$(7)\ 4m^2 - 8im - 4 + (a^2 + b^2) = 0,$$

from which

$$(8)\ m = \pm i \pm 1/2 \sqrt{-(a^2 + b^2)}.$$

The plus or minus sign is used with i according as the point under consideration is (1,i) or (1, -i) respectively. It is evident then that the curve has a double point at each of the conjugate imaginary points. The directions of the curves at these points are imaginary

since the values of m involve i.

The asymptotes to these curves may be determined in the ordinary way or the method used in article 9, 10 and 11 may be substituted when it proves convenient. Suffice it to say that when the curves are constructed, the tangent line to the original curve where it crosses r is drawn. The intersection of this line with s and the direction of the lines which determine the vanishing point of the transformed curve are sufficient to fix the asymptote. Since this is a general statement for the cyclic curves it will not be repeated but their discussion will consist only of the results of the study of their singular points at infinity.

17. Pascal's Limaçon. Among the plane curves which Pascal was wont to study is one to which Giles Persone de Roberval has given the name Pascal's Limaçon. It is usually written in trigonometric form but by simple substitutions may be reduced to the algebraic form

$$(1)\quad (x^2 + y^2 + 2bx)^2 - 4a^2(x^2 + y^2) = 0.$$

By the transformation in the previous article the curve becomes

$$(2)\quad \left[x'^2 + y'^2 + 2bx'(x'-1)\right]^2 - 4a^2(x'-1)^2(x'^2+y'^2) = 0.$$

Again by substituting

$$(3)\quad y' = m(x'-1) \pm i,$$

which is the equation of the tangent to the curve at the conjugate imaginary points we obtain

$$(4)\quad \left[x'+1+m^2(x'-1) \pm 2im+2bx'\right]^2 - 4a^2(x'-1)\left[x'+1+m^2(x'-1) \pm 2im\right] = 0.$$

Now the limit as $x' \doteq 1$ is taken and the quadratic equation

$$(5)\quad (\pm im+b+1)^2 = 0,$$

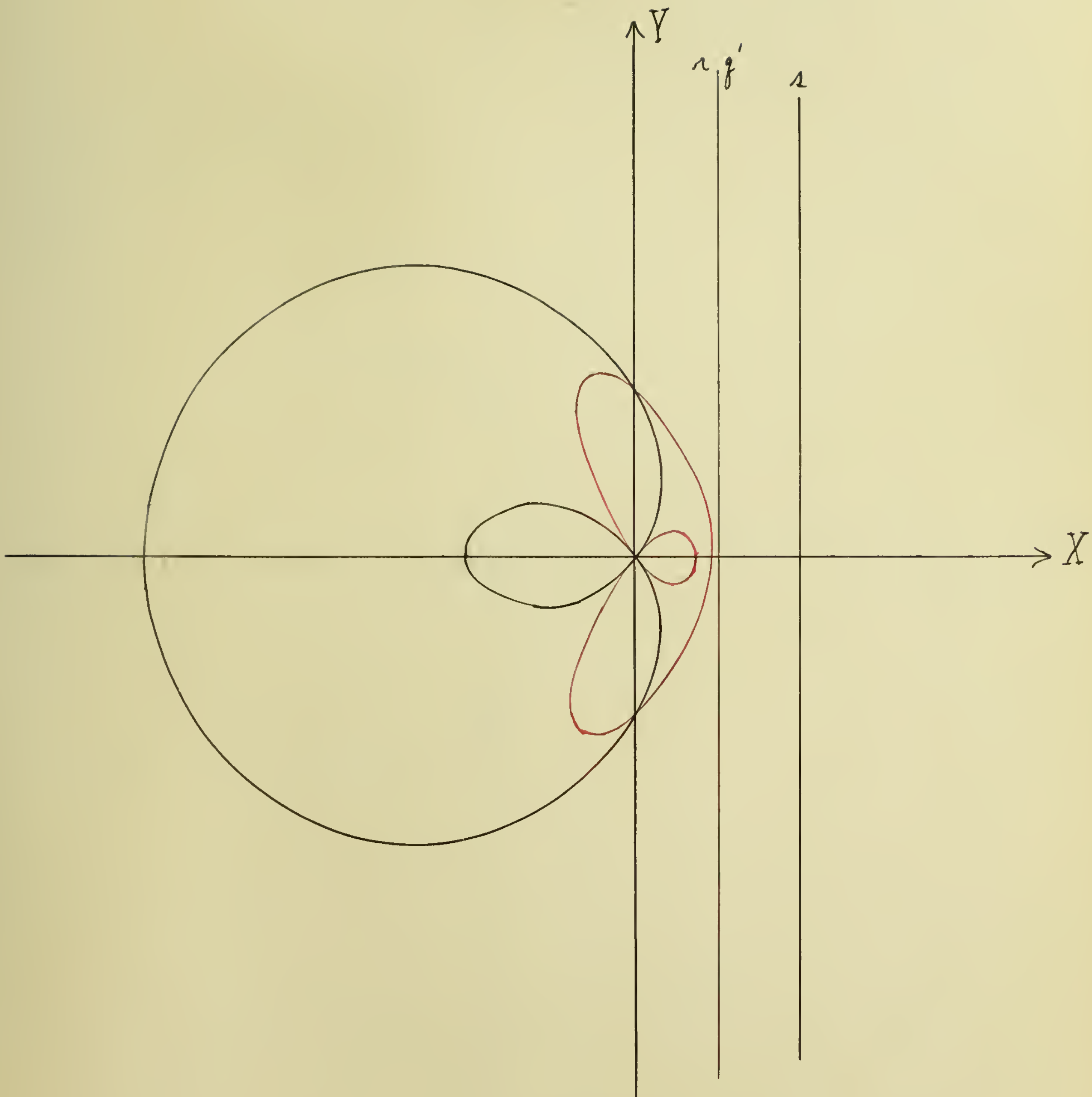

Fig. 19.

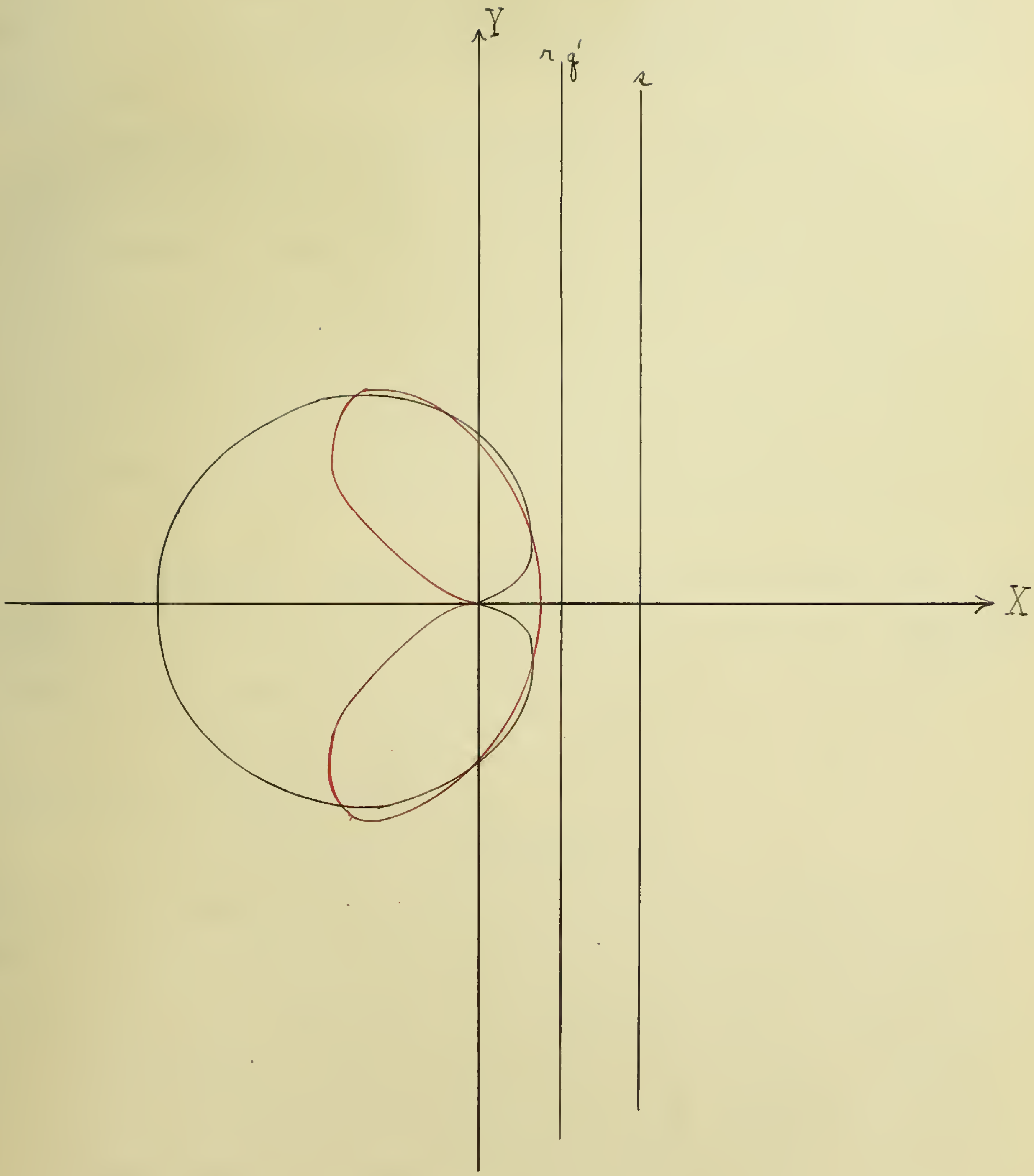

Fig. 20.

whence

$$(6)\ m = \pm\, i(b+1),$$

where the plus or minus sign for the i is used again according as the point under consideration has the y coordinate plus or minus i respectively.

From equation (5) we see that for i either plus or minus m has two values which are equal. The conclusion drawn from this fact by the aid of the previous study, is that the curve has a cusp at each of the conjugate imaginary infinite points.[1] However this does not coincide with the case in article 14 where p equals 1 and so we cannot say that the infinite line is the tangent to the curve at these points. The results of the construction are given in Fig. 19.

18. Cardioid. Figure 20 represents a curve which was studied in the eighteenth century and which is called the cardioid. Its equation in trigonometric form is

$$(1)\ \rho = 2a(1-\cos\phi),$$

but which can be easily reduced to

$$(2)\ (x^2+y^2+2ax)^2 - 4a^2(x^2+y^2) = 0.$$

Since the a^2 in the second term of this equation is found in the previous article to disappear and since the b in the first term of equation (2) of that article is replaced in this case by a, we need only substitute a for b in the value of m in order to determine that the two curves have the same kind of singularities at the conjugate imaginary points at infinity.[2]

19. Three Cusped Hypocycloid. The three cusped hypocycloid

(1) See G. Loria- Specielle Algebraische und Transcendente Ebene Kurven Theorie und Geschichte. Vol. I, p. 134.

(2) See G Loria- Specielle Algebraische und Transcendente Ebene Kurven Theorie und Geschichte. Vol. I. p. 144.

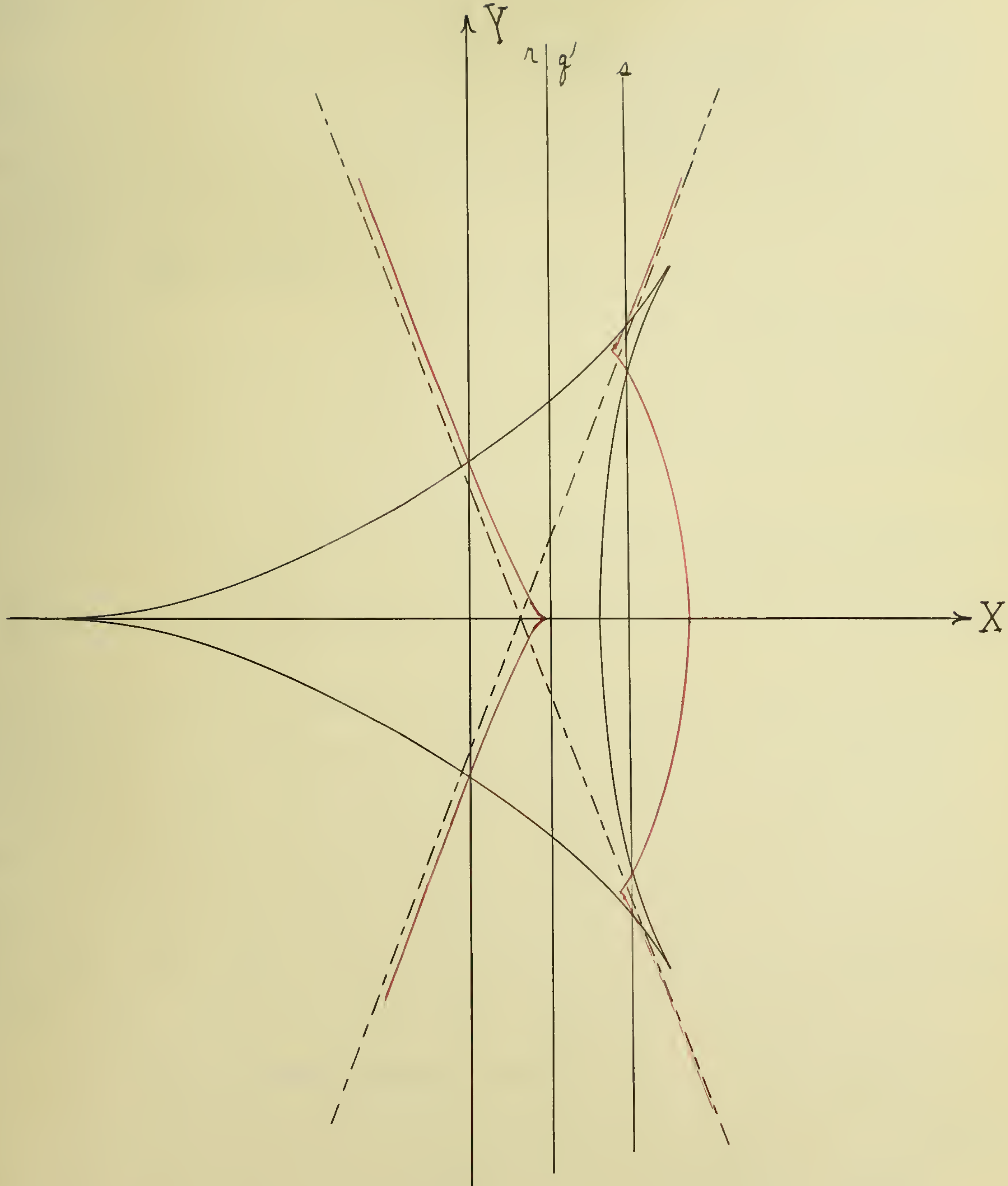

Fig. 21.

shown in Fig. 21 is an outgrowth of the study of the cycloid by Bernoulli and Pascal. Its equation is

$$(1)\quad (x^2+ y^2) + 4xr(3y^2-x^2)+\frac{9r^2}{2}(x^2+y^2) - 9/16 = 0.$$

Without loss of generality r may be taken equal to one and then after substituting the transformation given above the result reduces to

$$(2)\quad (x'^2+y'^2) + 4x'(x'-1)(3y'^2 - x'^2) + 9/2(x'-1)^2(x'^2+y'^2) - 9/16(x'-1)^4 = 0.$$

These curves are shown by the graphs of Fig. 21. Here again we solve the equation of the tangents through the cyclic points simultaneously with (2) and have as a consequence

$$(3)\quad (x'-1)^2\left[x'+1+m^2(x'-1)\pm 2im\right]^2 + 4x'(x'-1)\left[3m^2(x'-1)^2 \pm 6im(x'-1)-3-x'^2\right] + 9/2\,(x'-1)^3\left[x'+1+m^2(x'-1) \pm 2im\right] - 9/16\,(x'-1)^4 = 0.$$

It is easily seen from this equation that when x' equals one, the coefficients of the powers of m are zero and hence the values of m must be infinite. This means that the line joining the cyclic points is tangent, at each of these points, to both of the branches passing through that point. Moreover, since (x^2+y^2) is raised to the second power, the curve cannot cross the tangent but each branch at each point must have an ordinary tangency at the point.[1]

20. Three Phase Pendulum. Another curve which we shall discuss is one which arises from the study of the path of a compound pendulum. The three phase pendulum as it is called is usually written in the parametric form

$$(1)\quad x = r(2\cos 2\alpha - \cos\alpha),$$
$$y = r(2\sin 2\alpha - \sin\alpha).$$

[1] See G. Loria. Same volume as mentioned in previous article, p. 149.

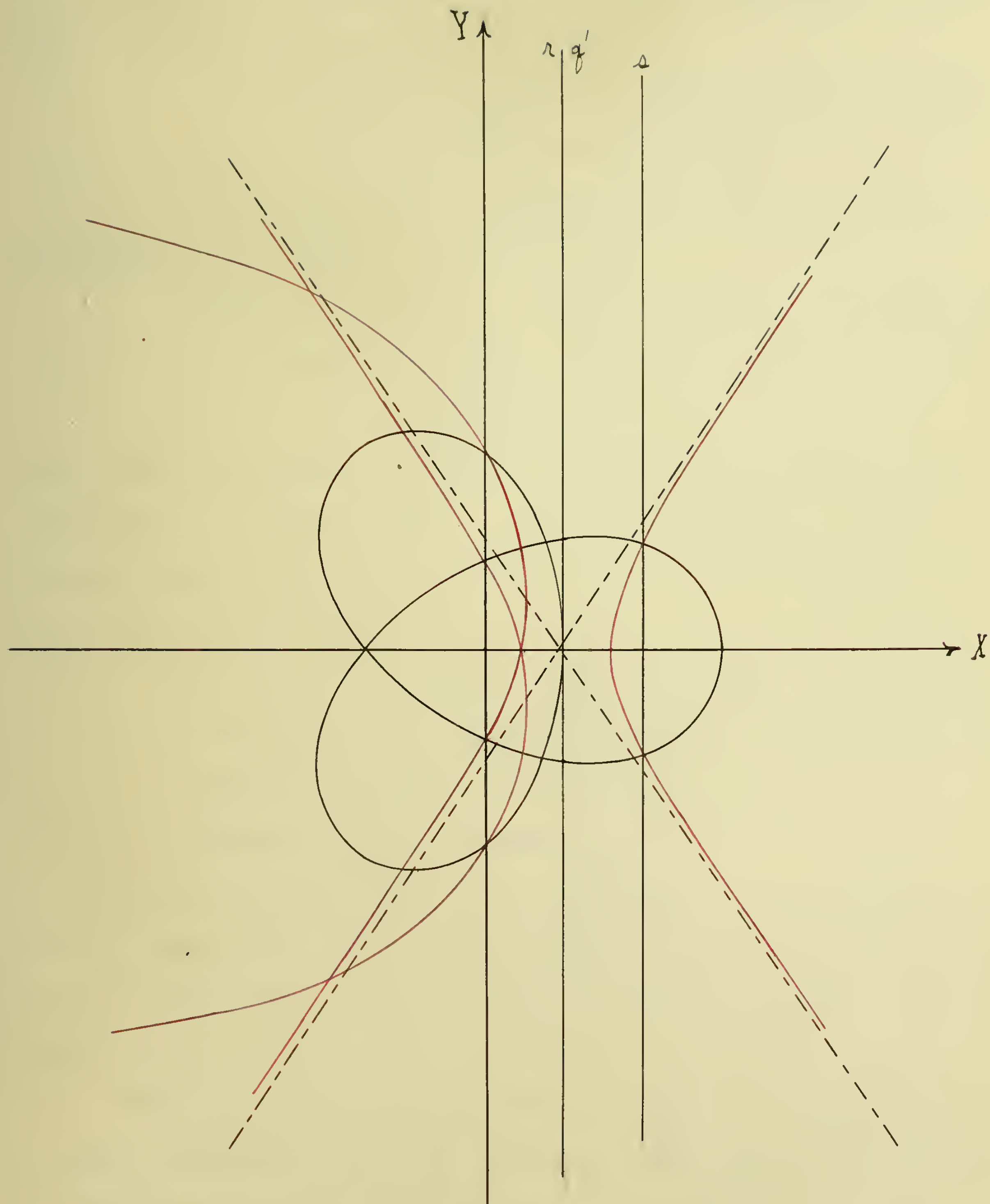

Fig. 22.

Eliminating the trigonmetric functions from these equations the curve in Cartesion coordinates is given by

$$(2)\ 4096(x^2+y^2)^2 + 4609rx^3 - 40123\, r^2x^2 + 7163rxy^2 - 38208r^2y^2 - 19824r^3x + 83088r^4 = 0.$$

By means of the transformation used in the previous articles of this chapter the equation takes the form

$$(3)\ 4096(x'^2+y'^2)^2 + 4609r(x'-1)x'^3 - 40123r^2(x'-1)^2x'^2 +7163r(x'-1)x'y'^2 - 38208r^2(x'-1)^2y'^2-19824r^3(x'-1)^3x' + 83088r^4(x'-1)^4 = 0.$$

These curves are shown in Fig. 22. Without loss of generality we may assume r equal to one. The equation of the tangent to the curve at the cyclic points is solved with (3) and the resulting equation in m and x' is

$$(4)\ 4096\left[x'^2 -1+m^2(x'-1)^2\pm 2im(x'-1)\right]^2 + 4609(x'-1)x'^3 -40123(x'-1)^2x'^2+7163(x'-1)x'\left[m^2(x'-1)^2 \pm 2im(x'-1)-1\right] -38208(x'-1)^2\left[m^2(x'-1)^2 \pm 2im(x'-1)-1\right] -19824(x'-1)^3x' + 83088(x'-1)^4 = 0.$$

As $x' \doteq 1$, it is evident that the coefficients of the powers of m become 0, and since $(x^2 + y^2)$ is raised to the second power, the results of this study coincide precisely with the results of the previous article.

We have shown now that a cyclic curve, i.e. one which has at least one factor $(x^2 + y^2)$ involving the highest terms, passes through the conjugate imaginary points at infinity. We have investigated their singularities by transforming these points to the conjugate imaginary points on the line $x' = 1$. The converse of this problem now presents itself. Is it possible to write a

curve through the conjugate imaginary points of the line $x = 1$, and after transforming it as before have a curve which has $(x'^2 + y'^2)$ as a factor in at least one term? This question is answered in the next article.[1]

21. $\left[(x-1)f(x,y) + (y^2 + 1)g(x,y)\right]^2 + \lambda(x-1)^n h(x,y) = 0.$

By Noethers Theorem[2] we may write the equation of a curve through the conjugate imaginary points of the line $x = 1$ in the form

$$(1)\quad \left[(x-1)f(x,y)+(y^2+1)g(x,y)\right]^2 + \lambda(x-1)^n h(x,y) = 0.$$

where f, g, and h are arbitrary polynomials in x and y, where λ is a constant and where n is one or two according as the curve is tangent to or has double points on $x = 1$, at the circular points. Applying the transformation,

$$(2)\quad x_1 = x'/z', \qquad y_1 = y'/z',$$

to the transformation

$$(3)\quad x = \frac{x_1}{x_1-1}, \qquad y = \frac{y_1}{y_1-1},$$

we have

$$(4)\quad x = \frac{x'}{x'-z'}, \qquad y = \frac{y'}{x'-z'}.$$

Transforming (1) by the use of (4) and simplifying the results we have two equations which may be written in the forms

$$(5)\quad (x'^2 + y'^2)^2 F(x',y') + Z'G(x',y') = 0,$$

$$(6)\quad (x'^2 + y'^2)^2 F(x',y') - Z'G(x',y') + z'^2 H(x',y') = 0,$$

according to whether n equals 1 or 2 respectively. In either case the equation contains the factors $(x'^2 + y'^2)$ involving the highest terms and hence the curves are cyclic curves. The answer to the question proposed

(1) See G. Loria etc., same volume as in the previous article, p. 156.

(2) See Pascal's - Repertorium der Höheren Mathematik. S-128.

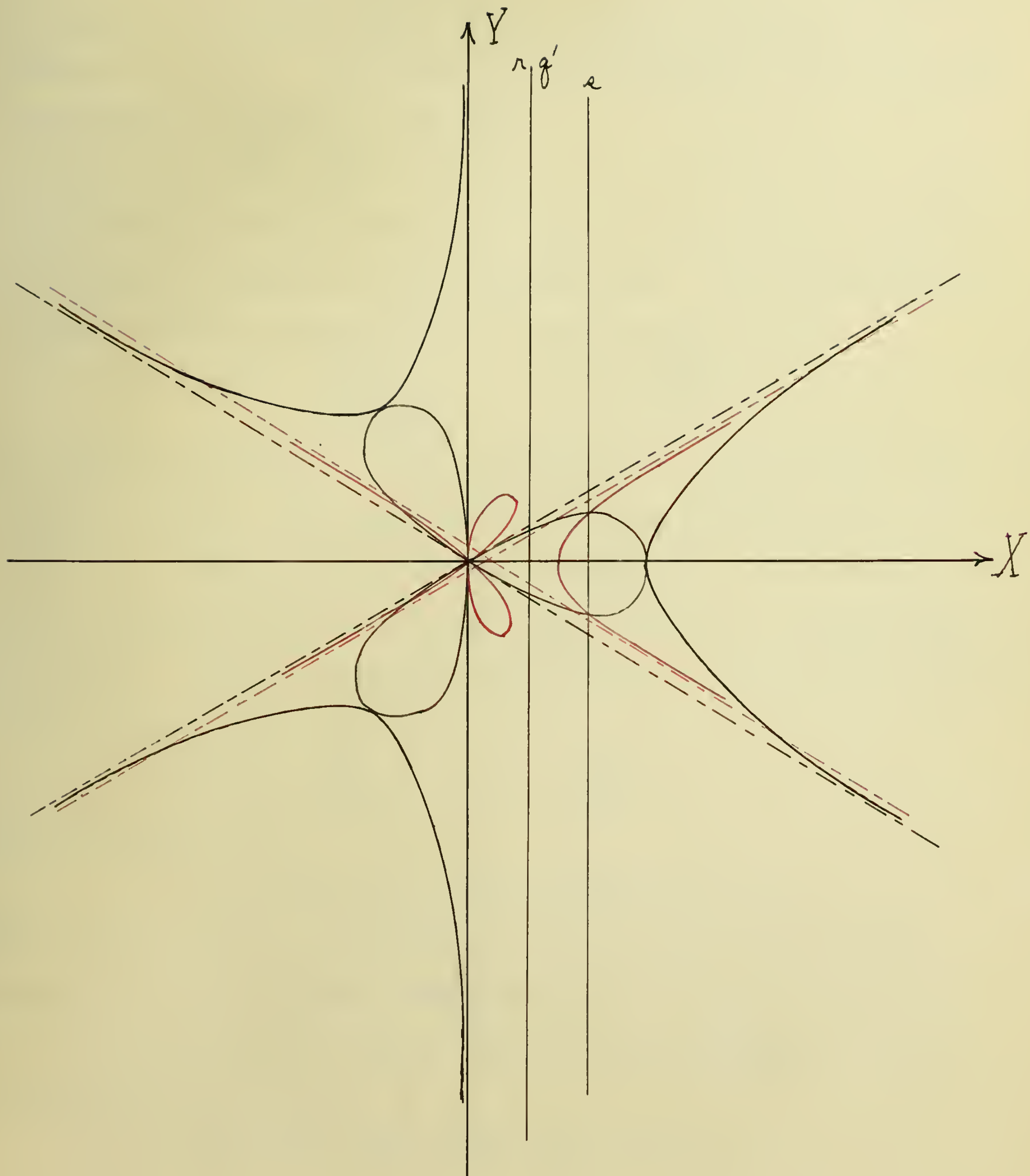

Fig. 23.

at the end of the previous article follows as a consequence of this result.

22. $x^3 - 3xy^2 - a^3 = 0.$

The equation of this elliptic cubic in polar coordinates is $\rho^3 \cos 3\theta - a^3 = 0$. By reducing it to Cartesian coordinates the curve has its equation in the form

$$(1)\ x^3 - 3xy^2 - a^3 = 0,$$

where a is the distance from the origin of the vertex of each of the asymptotic branches as shown in Fig. 23. Now if we utilize a special Cremona transformation[1] of the form

$$(2)\ x' = \frac{a^2x}{x^2 + y^2}, \quad y' = \frac{a^2y}{x^2 + y^2},$$

the cubic is transformed into a sextic of deficiency one. This will be proved later. The equation of the sextic reduces to

$$(3)\ (x^2 + y^2)^3 - a^3(x^3 - 3xy^2) = 0.$$

It is readily seen from the figure that there is a triple point at the origin. To find the form of the curve at infinity we transform (3) by the use of

$$(4)\ x = \frac{x'}{x'-1}, \quad y = \frac{y'}{x'-1},$$

and the result is

$$(5)\ (x'^2 + y'^2)^3 - a^3(x'-1)^3(x'^3 - 3x'y'^2) = 0.$$

When (5) is solved simultaneously with the equations of the tangents at the conjugate imaginary points of the line $x' = 1$, we have an equation in m which is of the third degree. To be precise, the equation is

$$(6)\ \pm 2im^3 + 6m^2 \mp 6im + 2 - a^3 = 0.$$

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(1) See Karl Doehlemann-Uber Cremona-Transformationen in der Ebene, welche eine Kurven enhalten, die sich Punkt fur Punkt selbst entspricht. Math. Ann. Bd 39, 1891, s567-697.

Clearly then, the sextic has triple points at infinity. Moreover since it also has a triple point at the origin and since the three triple points are equivalent to nine double points the curve is of deficiency one.

The conclusion is that a cubic of deficiency one when transformed by an involuntary quadratic transformation is changed into a sextic of deficiency one.

23. Conclusion. From this discussion of different curves by the various methods, it is evident that often one method is more convenient than another. Indeed for the binomial curves the method of homogeneous coordinates is preferable since it gives more briefly the desired results. Many devices have been invented to study the nature of curves at their infinite points. Among these are stereographic projection, reciprocal projection, projection by reciprocal radii and generally by Cremona's transformation. However, the methods used in this study are the simplest, and for the curves ordinarily encountered the results are obtained in a form more recognizable by the students of elementary mathematics than would be the case with the other methods.

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